

Properties of the maximal entropy measure and geometry of Hénon attractors

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Abstract

We show the existence and uniqueness of the maximal entropy measure for non-uniformly hyperbolic C^2 -Hénon like diffeomorphisms. This follows mostly from a geometrical study of the attractor and a conjugacy of a subset with a strongly positive recurrent Markov shift. Moreover a coding of the periodic points shows that the maximal entropy measure is equidistributed on them. The maximal entropy measure is also shown to be finitarily Bernoulli, exponentially mixing and satisfying the central limit Theorem.

We are going to study the ergodic properties of a large set of non uniformly hyperbolic diffeomorphisms of the form:

$$f_{a,B} : (x, y) \mapsto (x^2 + a + y, 0) + B(x, y, a),$$

where $B \in C^2(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2)$ is uniformly C^2 -close to 0. We denote by b an upper bound of the uniform C^2 -norm of B . In [Ber11] the following result is shown:

Theorem 0.1. *For any $\eta > 0$, for any a_0 greater but sufficiently close to -2 , there exists $b > 0$ such that for any B with C^2 -norm less than b , there exists a subset $\Omega_B \subset [-2, a_0]$ such that $\frac{\text{Leb } \Omega_B}{\text{Leb } [-2, a_0]} > 1 - \eta$ and for every $a \in \Omega_B$, the diffeomorphism $f_{a,B}$ is strongly regular.*

The definition of strong regularity is recalled in section 2. We showed in [Ber11] that this implies, for each $a \in \Omega_B$, the existence of a physical, SRB probability, left invariant by $f_{a,B}$; the associated attractor is non-uniformly hyperbolic.

Strongly regular maps enjoy of many properties. Their structure enables us to prove here:

Theorem 0.2 (Main result). *Every strongly regular map f leaves invariant a unique probability of maximal entropy ν . Moreover ν is equidistributed on the periodic points of f , is finitarily Bernoulli, is exponentially mixing and satisfies the central limit Theorem.*

Let $Fix f^p$ denotes the set of fixed points of f^p , and for $x \in \mathbb{R}^2$, let δ_x be the Dirac measure at x . A probability ν is *equidistributed on the periodic points of f* if the following limit converges to μ in the weak topology:

$$\frac{1}{\text{Card } Fix f^p} \sum_{x \in Fix f^p} \delta_x \rightharpoonup \mu, \quad p \rightarrow +\infty.$$

A *Bernoulli shift* is the shift dynamics of $\Sigma_N := \{1, \dots, N\}^{\mathbb{Z}}$ endowed with the product probability $p^{\mathbb{Z}}$ spanned by a probability $p = (p_i)_{i=1}^N$ on $\{1, \dots, N\}$. The entropy of the probability $p^{\mathbb{Z}}$ is $h_p = -\sum_i p_i \log p_i$. By Ornstein and Kean-Smorodinsky isomorphism Theorems, any two Bernoulli shifts $(\Sigma_N, p^{\mathbb{Z}})$ and $(\Sigma_{N'}, p'^{\mathbb{Z}})$ with the same entropy $h_p = h_{p'}$ are *finitarily isomorphic* [KS79]. This means that there exists a bi-measurable isomorphism sending $p^{\mathbb{Z}}$ to $p'^{\mathbb{Z}}$, such that the isomorphism and its inverse are continuous almost everywhere with respect to $p^{\mathbb{Z}}$ and $p'^{\mathbb{Z}}$.

To be *finitarily Bernoulli* means that the dynamics, with respect to the maximal entropy measure, is finitarily isomorphic to a Bernoulli shift.

The *central limit Theorem* is that for every Hölder function ψ of ν -mean 0, such that $\psi \neq \phi - \phi \circ f$ for any ϕ continuous, then there exists $\sigma > 0$ such that $\frac{1}{\sqrt{n}} \sum_{i=1}^n \psi \circ f^i$ converges in distribution (w.r.t. ν) to the normal distribution with mean zero and standard deviation σ .

The measure ν is *exponentially mixing* if there exists $0 < \kappa < 1$ such that for every pair of functions of the plane $g \in L^\infty(\nu)$ and h Hölder continuous, there is $C(g, h) > 0$ satisfying for every $n \geq 0$:

$$Cov_\nu(g, h \circ f^n) < C(g, h)\kappa^n, \quad \text{with } Cov \text{ the covariance.}$$

Introduction

Given a diffeomorphism f of a compact manifold M , it is fundamental to study of the *invariant* probabilities μ , (*i.e.* $f_*\mu = \mu$). Indeed, by Birkhoff ergodic theorem, for every Borelian subset U , a μ -generic point x has its orbit which lies in U in average as much as $\mu(U)$:

$$\frac{1}{N} \text{Card}\{n \leq N : f^n(x) \in U\} \rightarrow \mu(U), \quad N \rightarrow \infty,$$

whenever μ is *ergodic* (every f -invariant set has 0 or full μ -measure).

In general, a diffeomorphism leaves invariant many probabilities, and so we can wonder about which one should we regard. There are mostly two kinds of invariant measures with greater interests: those which are physical and those with maximal entropy.

An invariant probability μ is *Physical* if for every point x in a Lebesgue positive set B_μ , the Birkhoff sum $\frac{1}{N} \sum_{i=0}^{N-1} \delta_{f^i(x)}$ converges in the weak topology to μ . For every hyperbolic attractor, the existence and uniqueness of a physical measure μ is well known; the basin B_μ is formed by Lebesgue almost every point in a neighborhood of the attractor. The unique abundant example of non-uniformly hyperbolic (surface) attractor is given by a Lebesgue positive set of parameters of the Hénon (like) family ([BC91], [YW01], [Ber11]). For such dynamics, Benedicks and Young proved the existence and uniqueness of the (SRB) physical measure μ [BY93], Benedicks and Viana [BV01] showed that the basin of such a measure is Lebesgue almost every point in a neighborhood

of the attractor. Ergodic, physical measures are important from their mere definition. However, physicality is not invariant by C^0 -conjugacy: if g is conjugated to f by a homeomorphism h , then $h_*\mu$ is invariant by g but, in general, it is not anymore a physical measure. In particular, physical measures are not ergodically defined.

On the other hand, the entropy of a diffeomorphism f is invariant by conjugacy. Let us recall the definition of *topological entropy*. For two covers \mathcal{O} and \mathcal{O}' of M , the family of intersections of a set from \mathcal{O} with a set from \mathcal{O}' form a covering $\mathcal{O} \vee \mathcal{O}'$, and similarly for multiple covers. For any finite open cover \mathcal{O} of M , let $H(\mathcal{O})$ be the logarithm of the smallest number of elements of \mathcal{O} that cover M . The following limit exists:

$$H(\mathcal{O}, f) = \lim_{n \rightarrow \infty} \frac{1}{n} H(\mathcal{O} \vee f^{-1}\mathcal{O} \vee \dots \vee f^{-n}\mathcal{O}).$$

The *topological entropy* $h(f)$ of f is the supremum of $H(\mathcal{O}, f)$ over all finite covers \mathcal{O} of M .

Given a measure μ , the *entropy of μ* is defined similarly. For a finite partition \mathcal{O} , put:

$$H_\mu(\mathcal{O}, f) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{E \in \mathcal{O} \vee f^{-1}\mathcal{O} \vee \dots \vee f^{-n}\mathcal{O}} -\mu(E) \log \mu(E).$$

Then the *entropy h_μ of μ* is the supremum of $H_\mu(\mathcal{O}, f)$ over all possible finite partitions \mathcal{O} of M .

From the Variational Principle, the topological entropy is the supremum of entropies of invariant probabilities:

$$h(f) = \sup\{h_\mu(f) : \mu \text{ probability } f\text{-invariant}\}.$$

Therefore the topological entropy is also an *ergodic invariant*, *i.e.* it is invariant by bi-measurable conjugacy.

A probability μ has *maximal entropy* if $h(f) = h_\mu(f)$. In many cases, such a measure exists:

- If f is a C^∞ -diffeomorphism of surface, then there exists a maximal entropy measure [New89].
- If f is uniformly hyperbolic (Anosov or even Axiom A) then there exists a measure of maximal entropy [Par64], [Bow70].
- For every C^3 -perturbation of the Hénon family with small determinant, there exists a Lebesgue positive set of parameters for which the dynamics is non-uniformly hyperbolic and leaves invariant a probability of maximal entropy [YW01], (Cor. 10.1).

However a measure of maximal entropy needs not exist [Gur69].

The uniqueness of the maximal entropy probability is then fundamental: it gives a canonical measure of an ergodic system. In many cases, the maximal entropy measure is unique and of considerable interest:

- If f is a transitive Anosov [Par64], [Bow70],
- If f is a rational function of the Riemannian sphere [Mañ83], [Lju83],
- If f is a transitive map of the interval with positive entropy [Hof81],

- If f is a polynomial automorphism of \mathbb{C}^2 [BLS93].

The present work proves that for every C^2 -perturbation of the Hénon family with small determinant, there exists a Lebesgue positive set of parameters for which the dynamics is non-uniformly hyperbolic and leaves invariant a unique probability of maximal entropy. This result is stronger than the one of Young-Wang with respect to the following aspects: it deals with lower regularity and it provides both existence and uniqueness. Moreover it shows the aforementioned properties of the measure (equidistribution of periodic points, finitarily Bernoulli, exponentially mixing, central limit theorem).

Aside these motivations, the techniques used to prove this result are mostly geometric, and unable us to understand better the geometry and the dynamics of the Hénon attractor: existence of a Markov partition on a regular set and small Hausdorff dimension of irregular sets are shown.

We recall that the Hénon attractor is the only actual example of non uniformly hyperbolic attractor for (surface) diffeomorphisms: it is the only known to be abundant for generic families in the parameter space. Also Hénon like families are local models for homoclinic tangency unfoldings [PT93]; and Palis conjecture states these unfoldings typical for surface diffeomorphisms families which are not Axiom A [Pal08]. Therefore we hope this work would be basic to set up the thermodynamical formalism, not only for the Hénon family but also for most surface diffeomorphisms.

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1 Structure of the proof

1.1 Invariant splitting

Any strongly regular map f has a maximal invariant compact set Λ that we will split into three invariant subsets:

- the union of a fixed point A' with the intersection $\Lambda \cap W^s(A)$, where $W^s(A)$ is the stable manifold of another fixed point A ,
- the set of eventually infinitely regular points,
- the set of infinitely irregular points.

These invariant subsets are defined thanks to a family of partitions in §3, the family of all the pieces being indexed by a countable alphabet \mathfrak{A} .

Eventually infinitely regular points lie in the preimage of a set $\tilde{\mathcal{R}}$. In Corollary 3.5, we will see that $\tilde{\mathcal{R}}$ is homeomorphic to a product $\tilde{R} \times [0, 1]$, such that $\{\underline{a}\} \times [0, 1]$ corresponds to a “long stable manifold” for every $\underline{a} \in \tilde{R}$. The set \tilde{R} will be defined as a subset of $\mathfrak{A}^{\mathbb{N}}$. The set $\mathfrak{A}^{\mathbb{N}}$ is canonically

endowed with the shift dynamics. The points of \tilde{R} which return infinitely many times in \tilde{R} , by the shift map of $\mathfrak{A}^{\mathbb{N}}$, form a subset $R \subset \tilde{R}$:

$$R := \{(a_i)_{i \geq 0} : (a_{i+N})_{i \geq 0} \in \tilde{R}, \text{ for infinitely many } N \geq 0\}$$

It follows from the definition that the points of R come back infinitely many times in R .

The set of *infinitely regular points* $\mathcal{R} \subset \tilde{\mathcal{R}}$ is the subset corresponding to $R \times [0, 1]$. Let $\mathcal{E}_0 \subset \tilde{\mathcal{R}}$ be the subset corresponding to $(\tilde{R} \setminus R) \times [0, 1]$.

In Proposition 3.11, we will see that if the support of an invariant probability is off $\{A, A'\} \sqcup \bigcup_{n \geq 0} f^{-n}(\tilde{\mathcal{R}})$, then it is included in a disjoint hyperbolic compact set K_{\square} .

The points of $K_{\square} \sqcup \bigcup_{n \geq 0} f^{-n}(\mathcal{E}_0)$ are called *infinitely irregular*. The points $\bigcup_{n \geq 0} f^{-n}(\mathcal{R})$ are called *eventually infinitely regular points*.

We remark that every f -invariant probability ν has its support included in:

$$\{A, A'\} \sqcup \bigcup_{n \geq 0} f^{-n}(\mathcal{R}) \sqcup K_{\square} \sqcup \bigcup_{n \geq 0} f^{-n}(\mathcal{E}_0)$$

Note that $\{A, A'\}$ and K_{\square} are f -invariant by definition, and as every point of \mathcal{R} returns to \mathcal{R} , the set $\bigcup_{n \geq 0} f^{-n}(\mathcal{R})$ is f -invariant. Thus an ergodic probability has its support included in $\{A, A'\}$, $\bigcup_{n \geq 0} f^{-n}(\mathcal{R})$, K_{\square} or in $\bigcup_{n \geq 0} f^{-n}(\mathcal{E}_0)$.

By Poincaré recurrence theorem, a probability with support in $\bigcup_{n \geq 0} f^{-n}(\mathcal{E}_0)$ must have its support in \mathcal{E} with $\mathcal{E} := \bigcap_{N \geq 0} \bigcup_{n \geq N} f^n(\mathcal{E}_0)$. A similar fact holds for \mathcal{R} , and so any f -invariant probability ν satisfies:

$$\text{Supp } \nu \subset \{A, A'\} \sqcup \bigcap_{N \geq 0} \bigcup_{n \geq N} f^n(\mathcal{R}) \sqcup K_{\square} \sqcup \mathcal{E}.$$

The ergodic properties of each of these sets will imply the ergodic property of the Hénon map.

1.2 Ergodic property of the splitting component

The main ergodic property in which we are interested is the entropy. We can study it componentwise with respect to the above splitting.

- 1 The pair $\{A, A'\}$ consists of two fixed points, the dynamics induced on it has zero entropy.
- 2 In Proposition 5.5, we will see that the invariant set K_{\square} is either empty or the dynamics restricted to it has a very small entropy.
- 3 In Proposition 5.8, we will see that every ergodic measure ν supported in \mathcal{E} has small entropy. This is a consequence of the following arguments. In Proposition 5.7, we prove that the support of ν has Hausdorff dimension small. By Young entropy formula [You82], since the differential of f is bounded, it comes that the entropy of ν is small.

Further informations about \mathcal{E} are given: its stable and unstable Hausdorff dimensions are shown to be small in Proposition 5.1 and Corollary 5.10.

4 Consequently, the set $\cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$ contains the invariant probabilities with large entropy. The set \mathcal{R} enjoys of a Markov partition $(\mathcal{R}_g, n_g)_{g \in \mathfrak{B}}$. This means that for every $g \in \mathfrak{B}$:

(a) \mathcal{R}_g is an union of long stable manifolds in \mathcal{R} . More precisely, there exists $R_g \subset R$ such that \mathcal{R}_g corresponds to the set $R_g \times [0, 1]$ and for any points x, y in a same leaf:

$$d(f^n(x), f^n(y)) \leq b^{n/2}, \quad \forall n \geq 0.$$

(b) $f^{n_g}(\mathcal{R}_g)$ is included in \mathcal{R} .

(c) For every $g' \in \mathfrak{B}$ such that $f^{n_g}(\mathcal{R}_g) \cap \mathcal{R}_{g'} \neq \emptyset$, then $f^{n_g}(\mathcal{R}_g)$ intersects every leaf of $\mathcal{R}_{g'}$.

These Markovian properties are shown in Proposition 4.4. The dynamics induced by the *symbolic space* is:

$$F : x \in \mathcal{R} \mapsto f^{n_g}(x), \text{ if } x \in \mathcal{R}_g.$$

Unfortunately, a point $x \in \mathcal{R}_g$ might come back to \mathcal{R} before n_g -iterations, for $g \in \mathfrak{B}$. In other words, the map F is not the first return map of f in R . However, in Proposition 6.1, we show that it is the case for the set $\tilde{\mathcal{R}} := \cap_{n \geq 0} F^n(\mathcal{R})$:

(d) For every $g \in \mathfrak{B}$, every $x \in \mathcal{R}_g \cap \tilde{\mathcal{R}}$, the first return time of x in $\tilde{\mathcal{R}}$ is n_g .

The latter property is new: it does not appear in the previous theoretical models ([You98], [PSZ10]). It is fundamental in this proof.

We recall that the measures with support in $\mathcal{E} \cup K_{\square} \cup \{A, A'\}$ have small entropy. Consequently, the “interesting” measures are supported by $\cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$. In Proposition 4.3, we show that they are actually supported by $O(\tilde{\mathcal{R}}) := \cup_{n \geq 0} f^n(\tilde{\mathcal{R}})$.

(e) For every ergodic invariant probability μ , the sets $O(\tilde{\mathcal{R}})$ and $\cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$ are equal μ -almost everywhere. The probabilities with support off $O(\tilde{\mathcal{R}})$ have low entropy.

Properties (a)-(b)-(c)-(d) are useful to prove in Proposition 4.8 that:

(f) The dynamics on $O(\tilde{\mathcal{R}})$ is conjugated to a countable Markov shift (σ, Ω_G) modulo a subset of entropy zero, via a conjugacy h whose inverse is Hölder:

$$\begin{array}{ccc} O(\tilde{\mathcal{R}}) & \xrightarrow{h} & \Omega_G \\ f \downarrow & & \downarrow \sigma \\ O(\tilde{\mathcal{R}}) & \xrightarrow{h} & \Omega_G \end{array}$$

By *countable Markov shift*, we mean that G is a graph with countably many vertexes V and (oriented) arrows $\Pi \subset V^2$. The set Ω_G consists of the words $(g_i)_i$ such that (g_i, g_{i+1}) is in Π for every i . In Proposition 4.9, we show:

(g) The graph G is connected and the shift σ mixing.

By conjugacy, the entropy of $f|_{O(\tilde{\mathcal{R}})}$ and (σ, Ω_G) are equal. Also up to a periodic point, σ and $f|_{O(\tilde{\mathcal{R}})}$ share the same ergodic properties.

Other geometrical properties on $\tilde{\mathcal{R}}$ such as the existence of long unstable manifolds, a local product structure and the regularity of these structures will be studied in §5.2 and 6.4.

1.3 Ergodic consequences

We can bound the (topological) entropy of the maps f in which we are interested.

Indeed, f is C^2 -close to the Chebichev map $f_{-2} : x \mapsto x^2 - 2$, since for $a = -2$ and $B = 0$, the dynamics of $f_{-2,0}$ leaves invariant the line $\mathbb{R} \times \{0\}$ and its restriction is f_{-2} . The Chebichev map has entropy $\log 2$ since it is semi-conjugated to the doubling angle map of the circle. Furthermore, the maximal invariant of the complement of a small neighborhood of the critical point 0 by f_{-2} is a uniformly hyperbolic compact set with topological entropy close to $\log 2$. Thus by hyperbolic continuation, f preserves a hyperbolic compact set of entropy close to $\log 2$ (when $a \approx -2$ and $B \approx 0$). This shows:

Lemma 1.1. *The entropy of f is greater than $\log 2 - \epsilon$, for ϵ small when a is close to -2 and b is small.*

Let us mention that in the case of the *actual* Hénon map (polynomial map of degree 2), it holds [FM89]:

$$0 \leq h(g) \leq \limsup_n \frac{\log |Per_n(g)|}{n} \leq \log 2$$

As the entropy of any probability supported by the complement of $O(\tilde{\mathcal{R}})$ is small, by linearity of the entropy function, given an f -invariant probability ν with entropy close to the one of f , the entropy of the induced probability $\nu|_{O(\tilde{\mathcal{R}})}$ is strictly higher if $\nu(O(\tilde{\mathcal{R}})) < 1$.

This implies that existence (and uniqueness) of the maximal entropy measure for f holds if and only if existence (and uniqueness) of the maximal measure for the restriction $f|_{O(\tilde{\mathcal{R}})}$ holds.

As $f|_{O(\tilde{\mathcal{R}})}$ is conjugated to (σ, Ω_G) off the stable manifold of a fixed point, it comes that:

Claim 1.2. *Existence (and uniqueness) of the maximal entropy measure of f and σ are equivalent properties.*

A first way to have both existence and uniqueness of the maximal entropy measure is to work with a stronger hypothesis: assume that f is of class C^∞ . We recall that Newhouse Theorem [New89] implies the existence of a maximal entropy measure for any smooth surface diffeomorphism. Then the following result gives the uniqueness of ν_0 :

Theorem 1.3 (Gorevitch [Gur70]). *Let $G = (V, \Pi)$ be a connected, countable oriented graph. Let $\mathcal{M}(\sigma)$ be the set of σ -invariant Borel probabilities of Ω_G .*

If $\bar{h}(\sigma) = \sup_{\mu \in \mathcal{M}(\sigma)} h_\mu$ is finite and if there exists μ_0 such that $h_{\mu_0} = \bar{h}(\sigma)$, then μ_0 is unique with this property.

Another way to proceed, without loss of generality by assuming smoothness, is to prove that the shift is *strongly positive recurrent*.

Let us define this concept for mixing shifts. For a certain symbol $e \in V$, let Z_n be the number of loops from e in the graph G and let Z_n^* be the number of loops from e in the graph G passing

by e only at the begin and the end. Note that $Z_n^* \leq Z_n$. Let $R \leq R_*$ be the respective radius of convergence of the following series:

$$\sum_n Z_n X^n, \quad \sum_n Z_n^* X^n.$$

The shift is strongly positive recurrent if $R < R_*$. From this follows a lot of properties as explained below.

As, in our case, the entropy $h_{top} = \bar{h}(\sigma)$ of the shift is finite, by [Gur69], the radius R is at most $e^{-h_{top}}$ and so is at most close to $1/2$. On the other hand, we will show in Proposition 4.10 that the convergence radius R_* is at least close to one. Therefore, the $R < R_*$ (actually $2R \lesssim R_*$) and the shift is indeed strongly positive recurrent.

We can summarize the classical properties of strongly positive recurrent shift as¹:

Theorem 1.4 (Cyr-Sarig, Thm. 1.1-2.1 [CS09]). *Let Ω_G be a topologically mixing countable Markov chain which is strongly positive recurrent and with finite topological entropy. Then there exists a unique maximal entropy probability; this measure satisfies the central limit theorem and is exponentially mixing.*

As the conjugacy h^{-1} is shown to be Hölder continuous in §6.4, this achieves the proof of the main theorem but the properties of finitarily Bernoulli and of equidistribution on the periodic points.

The shift being strongly positive recurrent, we can give an explicit expression of its maximal entropy measure. Let $M = (m_{ij})_{i,j \in V}$ be the transition matrix associated to G : $m_{ij} = 1$ if $[i, j] \in \Pi$ and 0 otherwise. By Theorem D of [VJ67], the matrix M has $e^{h_{top}}$ as a maximal eigenvalue. Moreover, up to a scalar multiplication, there exist a unique eigenvector $(\alpha_i)_{i \in V}$ and a unique eigencovector $(\beta_i)_{i \in V}$ associated to this eigenvalue, with nonnegative coordinates, satisfying:

$$\sum_{i \in V} \alpha_i \beta_i = 1, \quad M(\alpha_i)_i = e^{h_{top}}(\alpha_i)_i, \quad (\beta_i)_i M = e^{h_{top}}(\beta_i)_i.$$

Then the maximal entropy measure corresponds to the Markov chain on V with invariant measure $(\pi_i)_{i \in V}$ with $\pi_i := \alpha_i \beta_i$ and probability of going from state $i \in V$ to state $j \in V$ in one step is equal to $p_{ij} := e^{-h_{top}} M_{ij} \beta_j / \beta_i$.

Therefore the probability to be in the state $e \in V$ and to go back to e only at the n^{th} step is $\pi_e Z_n^* e^{-n \cdot h_{top}}$. This probability decreases exponentially fast with n , by strong positive recurrence. A mixing shift enjoys of the latter property (called *exponentially decaying return time*) iff it is finitarily Bernoulli [Rud82].

Let us prove the equidistribution of periodic points. First we will show in Proposition 7.1 that the number of fixed points by f^p with orbit which does not intersect $\tilde{\mathcal{R}}$ is less than $p e^{p/\sqrt{M}} + (M+1)2^{p/(M+1)}$, for a certain large integer M . Such a number is negligible with respect to the number of fixed points of f^p :

$$\limsup \frac{1}{p} \log \text{card } \text{Fix}(f^p) \geq \limsup \frac{1}{p} \log Z_p = h_{top}.$$

¹It is stated there for one-sided shifts, however well known tricks generalize this result to 2-sided shifts.

Thus the average sum of the Dirac measures over the periodic points is equal to the one over the periodic points which lie in the orbit of $\check{\mathcal{R}}$. As the orbit of $\check{\mathcal{R}}$ is conjugated to the Markov shift everywhere but on the stable set of a 2-periodic point, there is a bijection between their periodic points (except one). A consequence of the two latter facts is that the following converges weakly to 0:

$$\frac{1}{\text{Card } \text{Fix } f^p} \sum_{x \in \text{Fix } f^p} \delta_x - \frac{1}{\text{Card } \text{Fix } \sigma^p} \sum_{\underline{a} \in \text{Fix } \sigma^p} \delta_{h^{-1}(\underline{a})} \rightarrow 0, \quad p \rightarrow \infty.$$

As the Markovian shift is positive strongly recurrent, we can apply the following classical theorem:

Theorem 1.5 (Thm D, [VJ67]). *If σ is Mixing, Markov, strongly positive recurrent, then the following converges weekly to the maximal entropy measure, as $p \rightarrow \infty$:*

$$\frac{1}{\text{Card } \text{Fix } \sigma^p} \sum_{\underline{a} \in \text{Fix } \sigma^p} \delta_{\underline{a}}.$$

Moreover, $\frac{1}{p} \log(\text{Card } \text{Fix } \sigma^p)$ converges to the topological entropy of σ .

It implies that the maximal entropy measure of f is equidistributed on the periodic points.

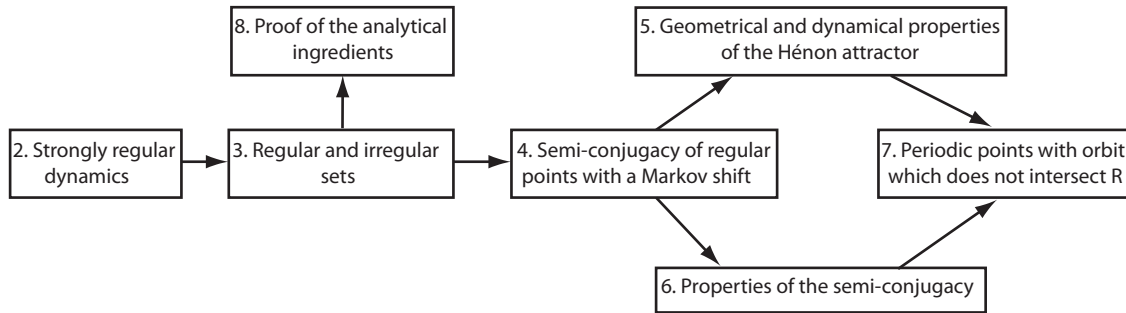


Figure 1: Interdependences of the sections.

At the end, an index gives the specific notations and definitions.

1.4 Open questions

Let us list a few questions which sounds reasonable to study after this work:

Question 1.6. *Does every strongly regular map enjoy of a unique equilibrium state for potentials of the form $s \cdot \log |\det Tf|_{W^u}|$?*

In this work, we answer positively to this question for $s = 0$. A way to answer to this question would be to generalize [CS09] to two-sided shift.

Question 1.7. *What is the Hausdorff dimension of the Hénon attractor?*

It is easy to show that the Lebesgue measure of the attractor is zero. We expect that the dimension should close to 1 for b small. From this work, it remains only to study the set of infinitely irregular points in the attractor.

Question 1.8. *In the full Hénon family of small Jacobean, does each parameter support a finite number of maximal entropy probabilities?*

Perhaps it is possible to combine the presented work with [Sar] and [Hof81], to answer to this question.

2 Strongly regular dynamics

In this section, we recall the definition of strongly regular dynamics and several ingredients useful for the aforementioned properties. For more details the reader is invited to read the first part of [Ber11].

2.1 Geometric model

When B is 0, the map $f_{a,B}$ equals $f_{a,0} : (x, y) \mapsto (x^2 + a + y, 0)$. The plane is sent by $f_{a,0}$ into the line $\mathbb{R} \times \{0\}$ and the restriction of $f_{a,0}$ to this line is the quadratic map $f_a : x \mapsto x^2 + a$. For a greater but close to -2 , the quadratic map f_a has two fixed points $-1 \approx A_0 < A'_0 \approx 2$ which are hyperbolic. Their respective hyperbolic continuations for B small are the fixed points A and A' of $f_{a,B}$.

We denote by b an upper-bound of the C^2 -norm of B and of the determinant of $Tf_{a,B}$. We put $\theta := 1/|\log b|$ and $c = \log 2/2$.

All the points of $[-A'_0, A'_0]$ are sent by an iterate of f_a either in A'_0 or in $[A_0, -A_0]$. Therefore all the points of the plane, which do not escape to infinity, are sent by an iterate of the dynamics $f_{a,B}$ either in the stable manifold of A' or into the compact domain Y_e bounded by:

- Two segments $\partial^s Y_e$ of the $f_{a,B}$ -stable manifold of A , given by the C^2 -persistence of the two segments $\{(x, y); f_a(x) + y = A_0, |y| \leq 2\theta\}$ of the $f_{a,0}$ -stable manifold of $A_0 \times \{0\}$. The set $\partial^s Y_e$ is formed by two connected curves with end points in $\partial^u Y_e$.

The set Y_e is diffeomorphic to the filled square $[0, 1]^2$, the boundary of Y_e is $\partial^u Y_e \cup \partial^s Y_e$.

We denote by U the 3θ -neighborhood of $[a, f_a(a)] \times \{0\}$. The orbit of U is included in U .

We denote by M the minimal iterate such that $f_a^M(a)$ belongs to $[A_0, -A_0]$; M is large since $a > -2$ is close to -2 . In the strongly regular definition, $f_a^M(a)$ will be supposed to do not be too close (in function of M) of $\{A_0, -A_0\}$.

Then for b small (in function of M large), the box Y_e is the union of boxes $(Y_s)_{s \in \mathfrak{Y}_0}$ and Y_\square .

By *box*, we mean that for every symbol $\alpha \in \mathfrak{Y}_0 \sqcup \{e, \square\}$, there exists a C^2 -diffeomorphism $y_\alpha^0 : [0, 1]^2 \rightarrow Y_\alpha$ such that:

- y_α^0 sends affinely $[0, 1] \times \{t\}$ onto a segment of $\mathbb{R} \times \{4\theta(t - \frac{1}{2})\}$, for every $t \in [0, 1]$. In particular, $\partial^u Y_\alpha := y_\alpha^0([0, 1] \times \{0, 1\})$ consists of two segments of the lines $\{y = 2\theta\}$ and $\{y = -2\theta\}$ respectively.
- $\partial^s Y_\alpha := y_\alpha^0(\{0, 1\} \times (0, 1))$ consists of two curves $\sqrt{b} - C^2$ -close to arc of parabolas $\{f_a(x) + y = cst\}$, where the constants cst are the same iff $\alpha = \square$.

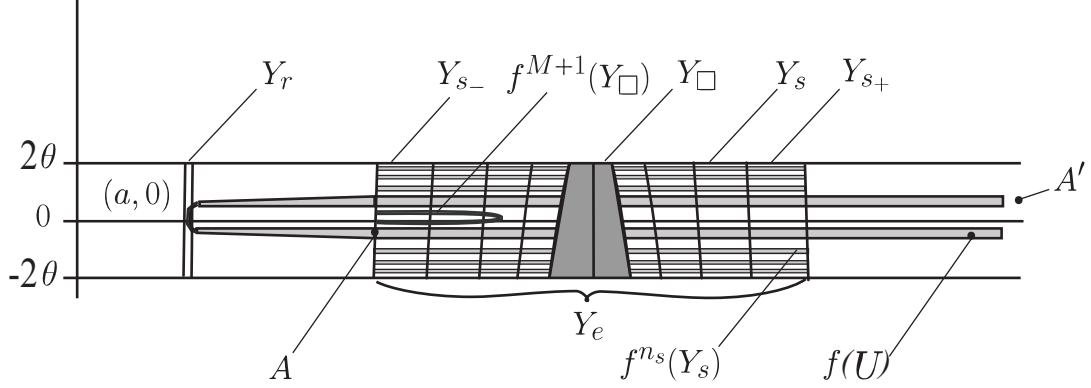


Figure 2: Geometric model for some parameters of the Hénon map.

The set $\mathfrak{Y}_0 \cup \{e, \square\}$ is formed by finitely many symbols α to which is associated not only a box Y_α and a diffeomorphism y_α^0 but also an integer n_α . Put $n_e = 0$ and $n_\square = M + 1$.

It satisfies the following conditions:

\hat{G}_0 Y_\square is a neighborhood of 0.

\hat{G}_1 For every symbol $s \in \mathfrak{Y}_0$, the set $f_{a,B}^{n_s}(Y_s)$ *stretches across* Y_e : this means that Y_s is sent into Y_e by $f_{a,B}^{n_s}$ and both components of $\partial^s Y_s$ are sent into exactly two components of $\partial^s Y_e$. The first return time of every $x \in Y_s$ in Y_e is n_s .

\hat{G}_2 The boxes $\{Y_s\}_{s \in \mathfrak{Y}_0, \{\square\}}$ have disjoint interiors and $Y_e = \cup_{s \in \mathfrak{Y}_0} Y_s \cup Y_\square$.

\hat{G}_3 The box Y_\square is sent into Y_e by $f_{a,B}^{M+1}$ and both components of $\partial^s Y_\square$ are sent into only one component of $\partial^s Y_e$. Moreover $M + 1$ is the first return time into Y_e of points in Y_\square .

Also the following analytical properties hold, with χ the cone field on \mathbb{R}^2 centered at $(0, 1)$ with angle $\theta := 1/|\log b|$:

\hat{G}_4 For every $s \in \mathfrak{Y}_0$, every $z \in Y_s$ and every $u \in \chi(z)$ satisfy, with $c := \log 2/2$:

$$\begin{cases} T f^{n_s}(u) \text{ belongs to } \chi(z), \\ \|T_z f_{a,B}^{n_s}(u)\| \geq e^{kc} \|T_z f_{a,B}^{n_s-k}(u)\|, \forall k \leq n_\alpha. \end{cases}$$

\hat{G}_5 For every $z \in Y_\square$, $\|T_z f_{a,B}^{M+1}(0,1)\| \geq e^{2Mc}$.

\hat{G}_6 The norms of Tf and T^2f on U are bounded by e^{c^+} , with $c^+ = \log 5$.

\hat{G}_7 The function $s \in \mathfrak{Y}_0 \mapsto n_s \in [2, M]$ is 2 to 1. Let $s_+, s_- \in \mathfrak{Y}_0$ be the two symbols such that $n_{s_\pm} = 2$. We suppose that Y_{s_-} contains the fixed point A .

The existence of such a model and a more detailed description is given in Proposition 1.7 [Ber11] (Hypothesis \hat{G}_7 is there given separately in §12.1). Actually, for every M large and then B C^2 -small, such a model exists. We recall that when M is large, the parameter a is close to -2 .

Remark 2.1. By \hat{G}_4 and \hat{G}_6 , for every $s \in Y_s$, the distance from Y_s to the line $\{0\} \times \mathbb{R}$ is greater than e^{-Mc^+} .

2.2 Flat and stretched curves settings

Let χ be the cone field on U centered at $(1,0)$ and with radius (in radian) less than θ .

A *flat curve* S is a C^{1+Lip} curve satisfying:

- S is included in $Y_e \cap \mathbb{R} \times [-\theta, \theta]$,
- the tangent space of S belongs to χ ,
- There exists an interval I_S of $[0, 1]$ such that S is the image by y_e^0 of the graph of a function $\rho \in C^{1+Lip}(I_S, [0, 1])$ whose derivative has Lipschitz constant less than θ .

Note that the end points of S belong to $\partial^s Y_e$.

A *flat stretched curve* is a flat curve such that I_S is equal to $[0, 1]$.

We endow the set of flat stretched curves with the distance:

$$d(S, S') = \max_{x \in [0, 1]} \|\rho(x) - \rho'(x)\| + \|d\rho(x) - d\rho'(x)\|,$$

where $S = y_e^0(\text{graph}(\rho))$ and $S' = y_e^0(\text{graph}(\rho'))$. The space of flat stretched curves is complete.

2.3 Puzzle pieces

The puzzle pieces are always associated to a flat, stretched curve S .

Definition 2.2. A (*hyperbolic*) *puzzle piece* α of S is the data of:

- an integer n_α called the *order of a puzzle piece* of α ,
- a segment S_α of S sent by f^{n_α} to a flat stretched curve S^α .

such that the following conditions hold:

h-times For every $z \in S_\alpha$, $w \in T_z S_\alpha$ and every $l \leq n_\alpha$: $\|T_z f^{n_\alpha}(w)\| \geq e^{\frac{c}{3}(n_\alpha-l)} \cdot \|T f^l(w)\|$.

In [Ber11] Expl. 2.2, we show:

Example 2.3 (Simple pieces). For any flat stretched curve S , each pair $s(S) := \{Y_s \cap S, n_s\}$, for $s \in \mathfrak{Y}_0$ or $e(S) := \{S, 0\}$ is a puzzle piece. On the other hand, $\{S \cap Y_\square, n_\square\}$ is not a puzzle piece. The pieces $\{s(S)\}_{s \in \mathfrak{Y}_0}$ are called *simple*. When no ambiguity is possible, we may write s instead of $s(S)$.

The following result is fundamental although elementary:

Proposition 2.4. *As the differential of f is bounded on U , for every $n > 0$, the number of puzzle pieces of a flat stretched curve S of order less than n is finite.*

Moreover two different puzzle pieces β, β' of a same flat stretched curve S are such that the segments $S_\beta, S_{\beta'}$ are either nested or with disjoint interiors.

For $s \in \mathfrak{Y}_0$, let $D(s)$ be the set of all flat stretched curves. From the above example, the following graph transform is well defined:

$$s : S \in D(s) \mapsto S^s := f^{n_s}(S_s) \in D(s)$$

Proposition 2.5 (Prop 14.9 [Ber11]). *The map s is $b^{\frac{n_s}{4}}$ -contracting for the metric of flat stretched curves, for every $s \in \mathfrak{Y}_0$.*

Operation \star on puzzle pieces Let $\alpha := \{S_\alpha, n_\alpha\}$ and $\beta = \{S_\beta^\alpha, n_\beta\}$ be two puzzle pieces of S and $S^\alpha := f^{n_\alpha}(S_\alpha)$ respectively. We define the puzzle piece of S :

$$\alpha \star \beta := \{f^{-n_\alpha}(S_\beta^\alpha) \cap S_\alpha, n_\alpha + n_\beta\}.$$

2.4 Puzzle pseudo-group

We are going to generalize the following structure.

Let $T_0 := \{\cdots s_i \cdots s_0 : (s_j)_j \in \mathfrak{Y}_0^{\mathbb{N}}\}$ be the set of presequences in the alphabet \mathfrak{Y}_0 .

Claim 2.6 ([Ber11], Claim 2.10.). *There exists a family of flat stretched curves $\Sigma_0 = (S^t)_{t \in T_0}$ such that for every $t \in T_0$, the curve S^t is endowed with the simple puzzle pieces $\mathfrak{Y}_0(t) \approx \mathfrak{Y}_0$ and satisfies:*

- for all $t \in T_0$ and $s \in \mathfrak{Y}_0$, the curve corresponding to the concatenation $t \cdot s \in T_0$ satisfies:

$$S^{t \cdot s} = f^{n_s}(S_s^t) =: (S^t)^s.$$

A *puzzle pseudo-group* is the data of a family $\Sigma = (S^t)_{t \in T}$ of flat stretched curves S^t endowed with hyperbolic puzzle pieces $\mathcal{Y}(t)$ satisfying the following properties with $\mathcal{Y} := \bigsqcup_{t \in T} \mathcal{Y}(t)$:

- there exists a bijection $(t, \alpha) \in \mathcal{Y} \mapsto t \cdot \alpha \in T$ such that the curves $S^{t \cdot \alpha}$ and $(S^t)^\alpha$ are equal,
- for all $t \in T$, $\alpha, \beta \in \mathcal{Y}(t)$, the segments S_α^t and S_β^t have disjoint interiors in S^t .

Example 2.7 (Simple puzzle pseudo-group). The pair $(\Sigma_0, \mathcal{Y}_0 = \bigsqcup_{t \in T_0, s \in \mathfrak{Y}_0} s(S^t))$ is a puzzle pseudo-group called *simple*.

2.5 Common sequence of puzzle pieces

Let $\sharp := \cdots s_- \cdots s_-$ and let S^\sharp be the flat stretched curve equal to the half local unstable manifold of A . With $[x]$ the integer part of x , let us denote:

$$\aleph := i \in \mathbb{N} \mapsto \begin{cases} \left\lceil \frac{\log M}{6c^+} \right\rceil & \text{if } i = 0, \\ \left\lceil \frac{c}{6c^+}(i + M) \right\rceil & \text{otherwise.} \end{cases}$$

Definition 2.8. For $N \in [1, \infty]$, a *common sequence* of pieces is a sequence $(\alpha_i)_{i=1}^N$ of hyperbolic puzzle pieces which satisfies the following properties:

1. α_1 is a puzzle piece of S^\sharp and, for $N \geq i > 1$, α_i is a puzzle piece of the flat curve $(S^\sharp)^{\alpha_1 \star \cdots \star \alpha_{i-1}}$,
2. each piece is either simple or has its segment in Y_\square ,
3. the following inequality holds for every $N \geq j \geq 0$:

$$\sum_{l \leq j: n_{\alpha_l} \geq M+1} n_{\alpha_l} \leq e^{-\sqrt{M}} \sum_{l=1}^{j-1} n_{\alpha_j},$$

4. if $\alpha_{n+1}, \alpha_{n+2}, \dots$ and α_{n+k} are equal to s_- then $k < \aleph(n)$; if $\alpha_{n+1} = s^+$ and $\alpha_{n+2} = \alpha_{n+3} = \cdots = \alpha_{n+k+1}$ are equal to s_- then $k < \aleph(n)$.

The composition $\star_{j=1}^i \alpha_j := \alpha_1 \star \alpha_2 \star \cdots \star \alpha_{i-1} \star \alpha_i$ is called a *common product of depth i* and defines a pair $c_i =: \{S_{c_i}^\sharp, n_{c_i}\}$ called a *common piece*.

A *common piece of depth 0* is the pair equal to $c_0 := e = \{S^\sharp, 0\}$.

An immediate consequence of the third statement of the common piece definition is:

Proposition 2.9. *Every common product of depth i has an order less than Mi .*

The following Proposition associates to a common product c_i of depth i a *common box* $Y_{\underline{c}_i}$. Put $\epsilon = 1/\sqrt{M}$ and $c^- := c - \epsilon$.

Proposition 2.10 ([Ber11], Prop. 3.6). *Let $(\alpha_j)_j$ be a sequence of hyperbolic puzzle pieces which satisfies common sequence conditions 1-2-3. Then for every i , the piece $c_i := \star_{j=1}^i \alpha_j$ is well defined and satisfies the following properties:*

- Both end points of $S_{c_i}^\sharp$ enjoy of two stable manifolds $\partial^s Y_{\underline{c}_i}$ which link $\{y = -2\theta\}$ to $\{y = 2\theta\}$ and which are \sqrt{b} - C^2 -close to a segment of a curve of the form $\{f_a(x) + y = cst\}$.
- The two components of $\partial^s Y_{\underline{c}_i}$ are linked by two segments $\partial^u Y_{\underline{c}_i}$ of $\{y = -2\theta\}$ and $\{y = 2\theta\}$ respectively.
- The box $Y_{\underline{c}_i}$ bounded by $\partial^s Y_{\underline{c}_i} \cup \partial^u Y_{\underline{c}_i}$ is diffeomorphic to a square $[0, 1]^2$.
- For every $x \in Y_{\underline{c}_i}$, every $u \in \chi(x)$:

$$\begin{cases} e^{c^- n_{c_i}} \leq \|T_x f^{n_{c_i}}(u)\| \leq e^{c^+ n_{c_i}}, \\ e^{-cMk} \leq \|T_x f^k(u)\|, \quad \forall k \leq n_{c_i} \end{cases}$$

The latter we will be generalized (and proved) in Proposition 3.3 and Lemma 3.9 below.

Note that given a common sequence of pieces $c = (\alpha_j)_{j=1}^\infty$, the sequence of compact subsets $(Y_{\underline{c}_i})_i$ is decreasing and its limit $W_c^s := \cap_{i \geq 1} Y_{\underline{c}_i}$ is called a *common stable manifold* and satisfies:

Claim 2.11 ([Ber11], Claim 3.7). *Every common stable manifold W_c^s is a local stable manifold which is (\sqrt{b}) - C^{1+Lip} -close to a segment of a curve of the form $\{f_a(x) + y = cst\}$ with end points in $\{y = -2\theta\} \cup \{y = 2\theta\}$.*

2.6 Parabolic operations

Let us explain how to construct puzzle pieces on the subset $S_\square = S \cap Y_\square$ for a flat stretched curve S .

Let S^\square be the curve $f^{M+1}(S_\square)$. As f is b - C^2 -close to $f_{a,0}$ which have a constant second derivative, the map f^{M+1} is $e^{3(M+1)c^+} b$ - C^2 -close to $f_{a,0}^{M+1}$. As b is supposed small w.r.t. M , the curve S^\square is \sqrt{b} - C^2 -close to a quadratic curve $[A, f^{M+1}(0)] \cdot [f^{M+1}(0), A]$ which goes straight from A to $f^{M+1}(0)$ and returns straight to A .

Let us suppose that S^\square is tangent to W_c^s , with $c = (\alpha_i)_i$ a common sequence of pieces. By Claim 2.11, the curve S^\square is tangent to W_c^s at a unique point.

The fourth condition of common sequence's definition bounds from below the distance between the tangency point and the curves of $\partial^s Y_{\underline{c}_j}$ with $c_j := \star_{i=1}^j \alpha_i$.

By Proposition 2.10, both curves of $\partial^s Y_{\underline{c}_j}$ are \sqrt{b} - C^{1+Lip} -close to an arc of parabola $\{f_a(x) + y = cst\}$. As S^\square is C^2 -close to the curve $[A, f^{M+1}(0)] \cdot [f^{M+1}(0), A]$, we deduce that S^\square intersects $cl(Y_{\underline{c}_j} \setminus Y_{\underline{c}_{j+1}})$ at zero or two curves, for every $j \geq 0$.

If this intersection is formed by two curves, let $S_{\square-(c_j-c_{j+1})}$ and $S_{\square+(c_j-c_{j+1})}$ be the backward images of these curves by $f^{M+1}|_{S_\square}$, such that $S_{\square-(c_j-c_{j+1})}$ is at the left of $S_{\square+(c_j-c_{j+1})}$.

Let Δ be one of the two symbols $\square_\pm(c_j - c_{j+1})$.

We denote by $\Delta(S)$ the pair $\{S_\Delta, n_\Delta\}$, with $n_\Delta := M + 1 + n_{c_j}$. We call this pair a *parabolic piece (of depth j associated to S and c)*.

The pair $\Delta(S)$ is not a puzzle piece since $f^{n_\Delta}(S_\Delta)$ does not stretch across Y_e .

But for instance if there exists a flat stretched curve S^Δ which contains $f^{n_\Delta}(S_\Delta)$ and there exists a puzzle piece α of S^Δ such that S_α^Δ is included in $f^{n_\Delta}(S_\Delta)$, then the pair $\Delta(S) \star \alpha := \{f^{-1}(S_\alpha^\Delta) \cap S_\Delta, n_\Delta + n_\alpha\}$ is a “topological” puzzle piece of S (a little extra hypotheses is needed to satisfy h -times property).

Note that we extended the \star -product to this situation.

Actually, the construction of $\{S_\Delta, n_\Delta\}$ does not need $(\alpha_k)_{k \geq j+2}$ to be defined, and so we can formalize this operation by a map which is defined on a much wider set.

Definition 2.12. Let $c := (\alpha_i)_i$ be a common sequence of puzzle pieces. Let Δ be a symbol of the form $\square_\delta(c_j - c_{j+1})$, $\delta \in \{\pm\}$. Let $D(\Delta)$ be the set of the flat stretched curves S such that:

1. S^\square intersects the interior of exactly one of the following boxes:

$$\underline{Y_{c_{i+1} \star \star_1^{N(i+1)} s_-}} \quad \text{or} \quad \underline{Y_{c_{i+1} \star s_+ \star \star_1^{N(i+1)} s_-}}.$$

2. S^\square intersects $Y_{\mathcal{C}_i} \setminus Y_{\mathcal{C}_{i+1}}$.

The second condition is actually either always or never satisfied for the curves which satisfy the first one. Note that $D(\Delta)$ might be empty.

For $S \in D(\Delta)$, let S_Δ be the closure of the component of $S_\square \cap f^{-M-1}(Y_{\mathcal{C}_i} \setminus Y_{\mathcal{C}_{i+1}})$ at the left (resp. right) if $\delta = -$ (resp. $\delta = +$). We can now define:

$$\Delta : S \in D(\Delta) \mapsto \{S_\Delta, n_\Delta\}, \text{ with } n_\Delta := M + 1 + n_{\mathcal{C}_i}.$$

In Propositions 4.8 and 5.1 of [Ber11], we defined an algorithm to construct for every $S \in D(\Delta)$ a flat stretched curve S^Δ containing S^{n_Δ} and which enjoys of the following properties.

Proposition 2.13. *The map $\Delta : S \in D(\Delta) \mapsto S^\Delta$ is such that for every $S \in D(\Delta)$:*

- S^Δ is a flat stretched curve and S^Δ contains $f^{n_\Delta}(S_\Delta)$.
- The curve S^Δ is $b^{n_\Delta/4}$ -close to $(S^\#)^{c_i}$.
- The map Δ is $b^{\frac{n_\Delta}{4}}$ -contracting for the metric of flat stretched curves.

2.7 Symbolic formalism

Let (Σ, \mathcal{Y}) be a puzzle pseudo-group. Let $CS_j(\mathcal{Y})$ be the set of common sequences of puzzle pieces in \mathcal{Y} of depth $j \geq 0$. This set is finite for every j by Proposition 2.4. This implies that the set:

$$\mathfrak{A} := \mathfrak{Y}_0 \sqcup \{\square_\delta(c_i - c_{i+1}) : c \in CS_{i+1}(\mathcal{Y}), i \geq 0, \delta \in \{\pm\}\}$$

is a countable set of symbols.

Given a chain $\underline{a} = a_1 \cdots a_n \in \mathfrak{A}^n$, we can define:

- $n_{\underline{a}} := \sum_i n_{a_i}$,
- $D(\underline{a}) := D(a_1) \cap \mathbf{a}_1^{-1}(D(a_2)) \cap \cdots \cap (\mathbf{a}_{n-1} \circ \cdots \circ \mathbf{a}_1)^{-1}(D(a_n))$.
- $\mathbf{a} : S \in D(\underline{a}) \mapsto S^{\underline{a}} := a_n \circ a_{n-1} \circ \cdots \circ a_1(S)$.
- $\underline{a} : S \in D(\underline{a}) \mapsto (S_{\underline{a}}, n_{\underline{a}}) := a_1(S) \star a_2(S^{a_1}) \star \cdots \star a_n(S^{a_1 \cdots a_{n-1}})$.

A *suitable chain (of symbols)* is a chain $\underline{a} = a_1 \cdots a_n \in \mathfrak{A}^n$ from a flat stretched curve S is such that S belongs to $D(\underline{a})$ and $S_{\underline{a}}$ has cardinality greater than 1.

The chain of symbols $\underline{a} = a_1 \cdots a_n$ is *complete* if a_n belongs to \mathfrak{Y}_0 and *incomplete* otherwise.

The chain \underline{a} is *prime* if $n \geq 1$ and for every $k < n$, the chain $a_1 \cdots a_k$ is incomplete. In other words, the symbol a_k is not simple for every $k < n$.

For $\underline{a} \in \mathfrak{A}^n$ and $t \in T_0$, we denote by $t \cdot \underline{a}$, the presequence of $\mathfrak{A}^{\mathbb{Z}^-}$ finishing by \underline{a} and with the $-i^{th}$ -coordinate of t as $-(n+i)^{th}$ coordinate.

2.8 Puzzle algebra and strong regularity

A *puzzle algebra* $(\Sigma, \mathcal{Y}, \Sigma^\square, C)$ is the data of a puzzle pseudo-group $(\Sigma = (S^t)_{t \in T}, \mathcal{Y})$ equipped with a family of flat stretched curves $\Sigma^\square = (S^t)_{t \in T^\square}$ and with a family of common sequences of puzzle pieces $C = \{c(t) = (\alpha_i(t))_i; t \in T \sqcup T^\square\}$ such that the following conditions hold:

- (SR_0) Σ is the family of the curves of the form $(S^t)^{\underline{a}}$ among $t \cdot \underline{a}$ such that t is in T_0 and \underline{a} is a complete, suitable chains of symbols in \mathfrak{A} from S^t .
- (SR_1) S^{t^\square} is tangent to $W_{c(t)}^s$ for every $t \in T \sqcup T^\square$,
- (SR_2) Σ^\square is the family of the curves of the form $(S^t)^{\underline{a}}$ among $t \cdot \underline{a}$ such that t is in T_0 and \underline{a} is a incomplete, suitable chains of symbols in \mathfrak{A} from S^t .
- (SR_3) The puzzle pieces set $\mathcal{Y}(t)$ consists of all the pairs of the form $\underline{a}(S^t)$ with \underline{a} a prime, complete, suitable chain of symbols in \mathfrak{A} from S^t , for every $t \in T$.
- (SR_4) For every $t \in T \sqcup T^\square$, each puzzle piece of $c(t) = (\alpha_i(t))_i \in C$ belongs to \mathcal{Y} .

We say that f is *strongly regular* if there exists such a puzzle algebra $(\Sigma, \mathcal{Y}, \Sigma^\square, C)$.

For $\alpha \in \mathcal{Y}(t)$, the chain of symbols \underline{a} such that $\alpha = \underline{a}(S^t)$ is called the \mathfrak{A} -spelling of α .

The main result of [Ber11] (Theorem 0.1) is the following:

Theorem 2.14. *Every strongly regular map leaves invariant the SRB measure a non uniformly hyperbolic attractor. Moreover, strongly regular maps are abundant in the following meaning:*

For every η , there exists $b > 0$, such that for every B of C^2 norm less than b , there exists a subset of P_B of $[-2, -2 + \eta]$ such that for every $a \in P_B$, the map $f_{a,B}$ is strongly regular and $\text{Leb } P_B \geq \eta(1 - \epsilon)$.

Remark 2.15. To fix the idea, we will suppose (in particular) the following very rough inequalities: $M \geq 1000$ and $-\log b \leq \exp \exp M$.

They are sufficient for the new conditions given by this work.

3 Regular and irregular sets

For a strongly regular f with structure $\{\Sigma = (S^t)_{t \in T}, \mathcal{Y}, \Sigma^\square = (S^t)_{t \in T^\square}, C = (c(t))_t\}$, we are going to encode the dynamics thanks to a family of partitions $(\mathcal{P}(t))_{t \in T \sqcup T^\square}$. This encoding will define the regular and irregular sets.

3.1 Partition $\mathcal{P}(t)$ associated to $t \in T \sqcup T^\square$.

Let $t \in T \sqcup T^\square$ and let $c = c(t)$ be its associated common sequence by (SR_1) . We recall that $(Y_{\underline{c}_i})_i$ is a nested sequence of filled squares the intersection of which is the curve W_c^s .

Therefore $\{Y_{\underline{c}_i} \setminus Y_{\underline{c}_{i+1}}; i \geq 0\} \cup \{W_c^s\}$ is a partition of Y_e . Put:

$$Y_{\square(c_i - c_{i+1})} = cl\left(f^{-M-1}(Y_{\underline{c}_i} \setminus Y_{\underline{c}_{i+1}})\right) \cap Y_\square, \quad Y_{\square c} := f^{-M-1}(W_c^s) \cap Y_\square.$$

These sets have a very tame geometry:

Proposition 3.1. *The boundary of $Y_{\square(c_i-c_{i+1})}$ is formed by segments of the lines $\{y = \pm 2\theta\}$ and by arcs of curves $\sqrt{b}\text{-}C^2$ close to parabolas of the form $\{f_a(x) + y = cst\}$.*

For every $z \in Y_{\square(c_i-c_{i+1})}$, every $n \geq 0$, the following inequality holds:

$$(\mathcal{PCE}^{n_{c_i}+M+1}) \quad \|T_z f^k(0, 1)\| \geq e^{-Mc^+k}, \quad \forall k \leq n_{\square_{\pm}(c_i-c_{i+1})} := M + 1 + n_{c_i}.$$

Moreover $T_z f^{n_{c_i}+M+1}(0, 1)$ makes an angle less than θ with $(1, 0)$ and

$$(\mathcal{CE}^{n_{c_i}+M+1}) \quad \|T_z f^{n_{c_i}+M+1}(0, 1)\| \geq e^{c^-(n_{c_i}+M+1)}.$$

Proof. Inequalities $(\mathcal{PCE}^{n_{c_i}+M+1})$ and $(\mathcal{CE}^{n_{c_i}+M+1})$ are given by Proposition 14.2 of [Ber11]. The geometry of $Y_{\square(c_i-c_{i+1})}$ follows from Lemma 13.5 of [Ber11], as Proposition 3.6 of [Ber11] or as Proposition 3.3 will. \square

In figure 3, we draw all the possible topological shapes for $Y_{\square(c_i-c_{i+1})}$. From this, we remark that $Y_{\square(c_i-c_{i+1})}$ has one, two or three components. There is at most one component disjoint from S^t . If such a component exists, we call it $Y_{\square_b(c_i-c_{i+1})}$. There are one or two components which intersect S^t . If there are two components which intersect S^t , then there is one component at the left of the other. We call this component $Y_{\square_-(c_i-c_{i+1})}$. The component at the right of the other is called $Y_{\square_+(c_i-c_{i+1})}$. If there is only one component which intersects S^t , we call it $Y_{\square_a(c_i-c_{i+1})}$. In this case, we shall split $Y_{\square_a(c_i-c_{i+1})}$ into two components $Y_{\square_{\pm}(c_i-c_{i+1})}$.

Let Δ be the vertical line passing through the top of the arc of parabola which bounds from below $Y_{\square_a(c_i-c_{i+1})}$. The line Δ splits the set $Y_{\square_a(c_i-c_{i+1})}$ into two components. The one at the left (resp. right) of the other is denoted by $Y_{\square_-(c_i-c_{i+1})}$ (resp. $Y_{\square_+(c_i-c_{i+1})}$). We add $\Delta \cap Y_{\square_a(c_i-c_{i+1})}$ to $Y_{\square_-(c_i-c_{i+1})}$. Figure 4 depicts this splitting.

We remark that $\mathcal{P}(t) := \{Y_s; s \in \mathfrak{Y}_0\} \cup \{Y_{\square_{\delta}(c_i-c_{i+1})}; i \geq 0, \delta \in \{+, -, b\}\} \cup Y_{\square_c}$ is a *partition of Y_e modulo $W^s(A)$* . This means that $\mathcal{P}(t)$ is a covering of Y_e and every pair of different elements of $\mathcal{P}(t)$ have their intersection in $W^s(A)$.

The partition $\mathcal{P}(t)$ is associated to the element $t \in T \sqcup T^{\square}$, since $c := c(t)$ depends on t .

Let $\mathfrak{P}(t) := \mathfrak{Y}_0 \sqcup \{\square_{\delta}(c_i - c_{i+1}) : i \in \mathbb{N}, \delta \in \{\pm, b\}\} \sqcup \{\square_c\}$ be the set of symbols associated. The set $\mathfrak{P}(t)$ is countable. If $a \in \mathfrak{Y}_0$, we already defined an integer n_a .

Put $n_{\square_{\delta}(c_i-c_{i+1})} = M + 1 + n_{c_i}$ for $i \in \mathbb{N}$ and $\delta \in \{+, -, b\}$. Put $n_{\square_c} = \infty$.

We remark that $\mathfrak{P}(t) \supset \{a \in \mathfrak{A} : S^t \in D(a)\}$.

3.2 Regular points

Given a suitable sequence of symbols $g = (a_i)_{i=1}^n \in \mathfrak{A}^n$ from the curve $S^{\#}$, the symbol a_{i+1} belongs to $\mathfrak{P}(t \cdot a_1 \cdots a_i)$ for every $i \in [0, n)$. This leads us to consider the set of points $z \in Y_e$ such that $f^{n_{a_1} \cdots n_{a_i}}(z)$ belongs to $Y_{a_{i+1}}$ for every $i < n$. Nevertheless this set has in general a very wild geometry. It is not the case if the sequence is Ξ -regular, with $\Xi := e^{\sqrt{M}}$.

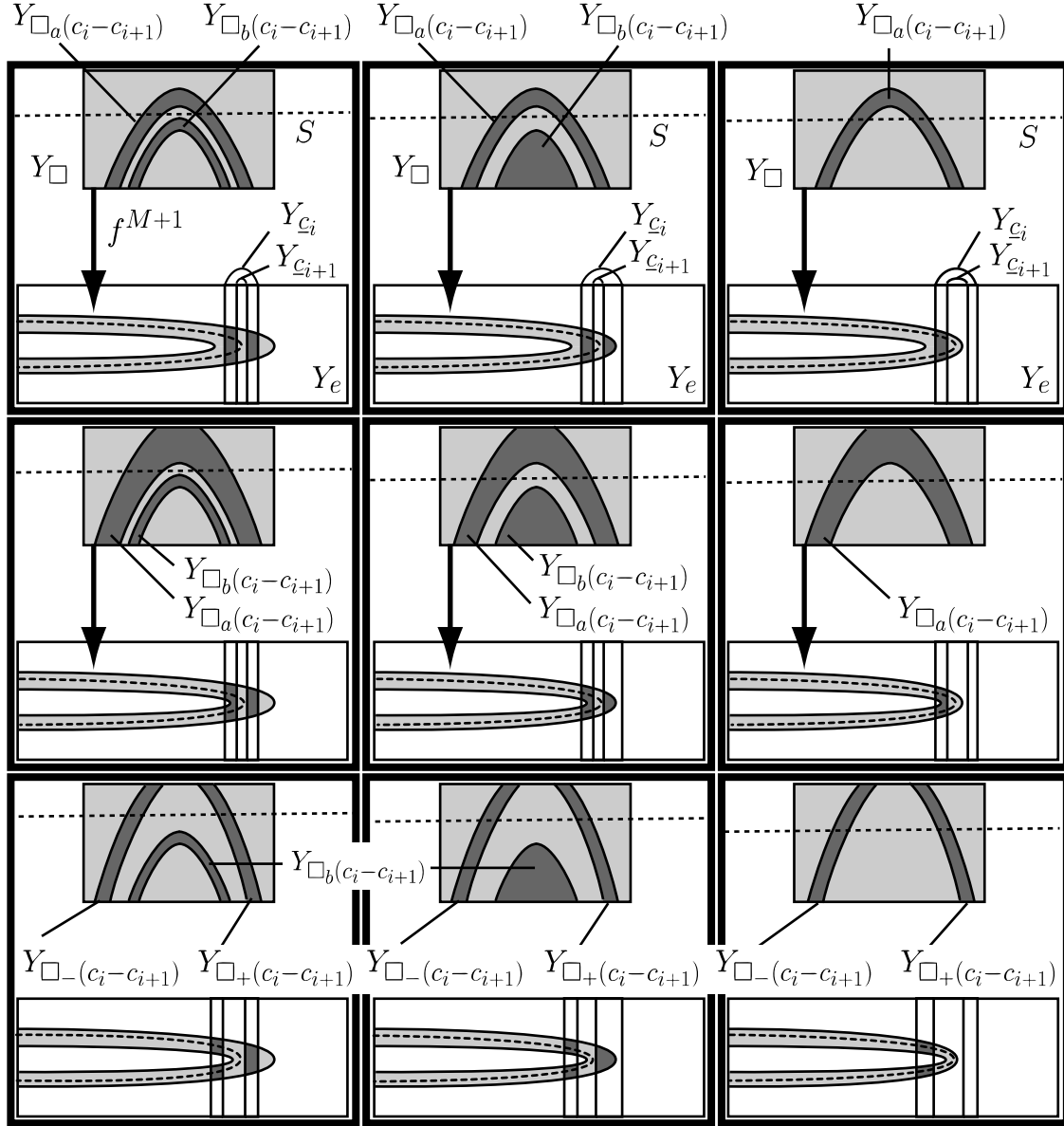


Figure 3: Possible shapes for $Y_{\square(c_i-c_{i+1})}$.

Definition 3.2. For every $\xi > 0$, a sequence of symbols $g = (a_i)_{i=1}^n \in \mathfrak{A}^n$ is ξ -regular if g is suitable from $S^\#$ and the following inequality holds for every $i \leq n$:

$$(R^\xi) \quad n_{a_i} \leq M + \xi \sum_{1 \leq j < i} n_{a_j}$$

We notice that a common sequence is Ξ^{-1} -regular. Also for $i = 1$, the above equation gives $n_{a_1} \leq M$ and so a_1 is simple, in other words $a_1 \in \mathfrak{Y}_0$.

Proposition 3.3. For every Ξ -regular sequence $g = a_1 \cdots a_n$, the set

$$Y_g := \{z \in Y_e : f^{n_{a_1} \cdots n_{a_i}}(z) \in Y_{a_{i+1}}, \forall i < n\}$$

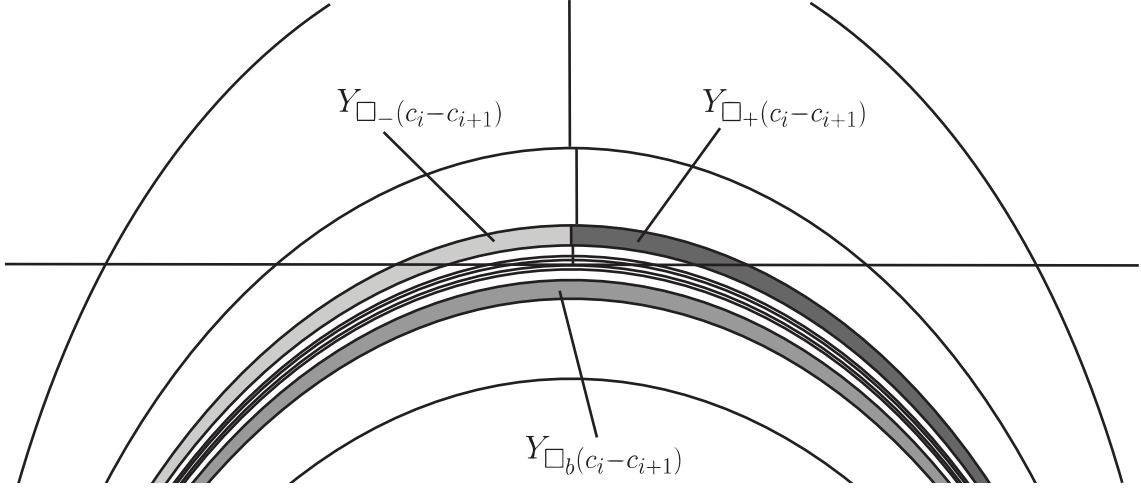


Figure 4: Partition of Y_{\square} .

is a box which satisfies the following properties:

- $\partial^s Y_g$ is formed by two segments of the stable manifold of A ; both link $\{y = -2\theta\}$ to $\{y = 2\theta\}$ and are $\sqrt{b}\text{-}C^2$ -close to an arc of a curve of the form $\{f_a(x) + y = cst\}$.
- Both components of $\partial^s Y_g$ are linked by two segments $\partial^u Y_g$ of $\{y = -2\theta\}$ and $\{y = 2\theta\}$ respectively.
- every horizontal line $\{y = C\}$, with $|C| \leq 2\theta$ intersects Y_g at a segment of length at most $e^{-cn_g/3}$.

The proof of this proposition is done in §8.

Remark 3.4. The same conclusions of Proposition 3.3 remain true if for suitable sequence $g =: (a_i)_i$ from $S^\#$ which satisfy:

$$(3.1) \quad n_{a_i} \leq \Xi(M + \sum_{j < i} n_{a_j}), \quad \forall i.$$

As for common sequences, given a Ξ -regular sequence $\underline{a} := (a_i)_i$, we can define $W_{\underline{a}}^s := \cap_{j \geq 0} Y_{a_1 \dots a_i}$. An immediate consequence of the above Proposition is the following:

Corollary 3.5. *The set $W_{\underline{a}}^s := \cap_{j \geq 1} Y_{a_1 \dots a_i}$ is a connected curve with an end point in each of the lines $\{y = \pm 2\theta\}$, and $W_{\underline{a}}^s$ is $\sqrt{b}\text{-}C^{1+Lip}$ -close to a segment of a curve of the form $\{f_a(x) + y = cst\}$.*

We are now ready to encode the dynamics, with respect to these regular sequences.

For $z \in Y_e \setminus W^s(A)$, let $\underline{a}(z) := (a_i(z))_{1 \leq i < p}$ be the maximal Ξ -regular sequence of symbols in \mathfrak{A} such that $z \in Y_{a_1 \dots a_i(z)}$, for every $i < p \in [0, \infty]$. Note that if $z \in Y_{\square}$, then $p = 0$ and the sequence $\underline{a}(z)$ is empty. Otherwise $p \geq 1$ and $a_1(z) \in \mathfrak{Y}_0$.

This sequence is uniquely defined. Indeed, as z belongs to $Y_e \setminus W^s(A)$, by induction on $i < p$, $f^{n_{a_1} \cdots a_i}(z)$ belongs to $Y_e \setminus W^s(A)$. Therefore there exists a unique symbol $a_{i+1} \in \mathfrak{P}(\# \cdot a_1 \cdots a_i)$ such that z belongs to $Y_{a_{i+1}}$.

Definition 3.6. Such a point $z \in Y_e \setminus W^s(A)$ is p - Ξ -regular. If $p = \infty$, the point z is Ξ -regular. For $\xi < \Xi$, the point z is ξ -regular if $p = \infty$ and the sequence $\underline{a}(z)$ is ξ -regular.

We notice that every point of $Y_e \setminus W^s(A)$ is at least 0- Ξ -regular.

We show below the following important Proposition:

Proposition 3.7. *For every p -regular point $z \in Y_e \setminus W^s(A)$ such that $0 < p < \infty$, the symbol $d \in \mathfrak{P}(\# \cdot a_1 \cdots a_{p-1}(z))$ such that $f^{n_{a_1} \cdots a_{p-1}}(z)$ belongs to Y_d satisfies:*

$$n_d > M + \Xi \sum_{1 \leq j < p} n_{a_j}.$$

In particular, this implies that d is of the form $\square c$ or $\square_\delta(c_i - c_{i+1})$, and $f^{n_{a_1} \cdots a_{p-1}(z) + M + 1}(z)$ belongs to W_c^s or Y_{c_i} , with $n_{c_i} \geq \Xi n_{a_1 \cdots a_{p-1}}$. This proposition is fundamental since then $f^{n_{a_1} \cdots a_{p-1} + M + 1}(z)$ is at least $\Xi n_{a_1 \cdots a_{p-1}}/M$ -regular.

This is why, for every p -regular point z , with $0 \leq p < \infty$, we complement the sequence $(a_i(z))_{i < p}$ of $z \in Y_e \setminus W^s(A)$ by the following inductive way:

Put $a_p(z) = \square$ with $n_\square = M + 1$ and then inductively $a_{p+i}(z) := a_i(z')$, for $i \geq 0$, with $z' := f^m(z)$ and $m = \sum_{i \leq p} n_{a_i}(z)$.

Definition 3.8. A point z is *infinitely irregular* if the sequence $\underline{a}(z)$ takes infinitely many times the value \square . Otherwise it is called Ξ -eventually regular. More generally, z is ξ -eventually regular, for $\xi \leq \Xi$, if z is Ξ -eventually regular and there exists j such that $(a_{j+i}(z))_{i \geq 1}$ is ξ -regular.

Proof of Proposition 3.7. Let $g := a_1(z) \cdots a_{p-1}(z)$ be the sequence associated to the p regular point z . We recall that χ is the cone filed centered at $(1, 0)$ with radius $\theta = 1/|\log b|$. Let us use the following Lemma shown in §8:

Lemma 3.9. *For every Ξ -regular sequence $g = a_1 \cdots a_n$, the box Y_g satisfies the following property:*

1. *For every $z \in Y_g$, every $u \in \chi(z)$*

$$\begin{cases} e^{\frac{c}{3}(n_g - k)} \|T_z f^k(u)\| \leq \|T_z f^{n_g}(u)\| & \forall k \leq n_g, \\ e^{-c^+(M+1+\Xi k)} \leq \|T_z f^k(u)\| & \forall k \leq n_g. \end{cases}$$

2. *Every $z \in Y_g$ belongs to a curve $\mathcal{C} \subset Y_g$, of length less than 1, intersecting every flat stretched curve and being θ^k contracted by f^k for every $k \leq n_g$.*

3. *The set $Y^g := f^{n_g}(Y_g)$ is a box such that $\partial^u Y^g := f^{n_g}(\partial^u Y_g)$ is the disjoint union of two flat curves and $\partial^s Y^g := f^{n_g}(\partial^s Y_g)$ is made by two θ^{n_g} -small segments of $W^s(A)$ passing through the end points of $f^{n_g}(S_g^\#)$.*

Remark 3.10. The second conclusion of this lemma remains true for the words g considered in Remark 3.4.

Thus the point $z' := f^{n_g}(z)$ belongs to the θ^{n_g} -neighborhood of $S^{\sharp \cdot g}$. If z' belongs to an extension of the form $Y_{\square_b(c_i - c_{i+1})}$, this implies that $f^{M+1}(z')$ belongs to the component of $Y_{\mathcal{E}_i} \setminus Y_{\mathcal{E}_{i+1}}$ which does not intersect $S^{\sharp \square}$. From the tangency position, the curve $S^{\sharp \square}$ is $e^{-(2\aleph(i+1)+1+n_{c_{i+1}})c^+}$ -far from this other component. On the other hand, as z' is θ^{n_g} -close to $S^{\sharp g}$ and belongs to the convex Y_{\square} , the point z' is θ^{n_g} -close to $S_{\square}^{\sharp g}$. Thus $f^{M+1}(z')$ is $\theta^{n_g}e^{c^+(M+1)}$ -close to $S^{\sharp \square}$. Therefore, z' can belong to $Y_{\square_b(c_i - c_{i+1})}$ only if:

$$(3.2) \quad e^{-(n_{c_{i+1}} + 2\aleph(i+1)+1)c^+} \leq \theta^{n_g} e^{c^+(2M+1)}$$

As $(n_{c_{i+1}} + 2\aleph(i+1) + 1)c^+ \leq (2c^+ + \frac{c}{3})n_{\square_b(c_i - c_{i+1})}$, inequality (3.2) implies:

$$-(2c^+ + \frac{c}{3})n_{\square_b(c_i - c_{i+1})} \leq n_g \log \theta + 2c^+ n_{\square_b(c_i - c_{i+1})}.$$

As by Remark 2.15 $-\log \theta / (4c^+ + c/3) \geq \Xi + M$, it comes that $n_{\square_b(c_i - c_{i+1})}$ is greater than $(M + \Xi)n_g$.

If z' belongs to a box of the form Y_{Δ} with $\Delta = \square_{\pm}(c_i - c_{i+1})$ such that $n_{\Delta} \leq M + \Xi n_g$ and $\Delta \in \mathfrak{P}(\sharp \cdot g)$. Let us show that $g \cdot \Delta$ is regular. Actually we need only to prove that $g \cdot \Delta$ is suitable. We already know that $S^{\sharp \cdot g}$ belongs to $D(\Delta)$. Suppose for the sake of contradiction that $g \cdot \Delta$ is not suitable from S^{\sharp} and hence $S_1 := f^{n_g}(S_{\Delta}^{\sharp})$ does not intersect the interior of $S_2 := S_{\Delta}^{\sharp \cdot g}$. Then the segment S_2 is either at the left or at the right of S_1 . We recall that $f^{n_g}(z)$ belongs to the intersection of Y^g with Y_{Δ} . As Y^g is not included in Y_{Δ} (since Y^g intersects $S^{\sharp \cdot g}$ at a set with non empty interior), Y^g intersects the boundary of Y_{Δ} . By the above computation on the distance given by the tangency, the intersection cannot happen at $\partial^u Y_{\Delta}$. Thus Y^g intersects $\partial^s Y_{\Delta}$.

If $\partial^s Y^g$ does not intersect $\partial^s Y_{\Delta}$, then one of the curve of $\partial^s Y_{\Delta}$ stretches vertically across Y^g , and so intersects the interior of S_1 . This leads to a contradiction since this intersection point would be an end point of S_2 .

If $\partial^s Y^g$ does intersect $\partial^s Y_{\Delta}$, then as both are formed by a slice of $W^s(A)$, one component of $\partial^s Y^g$ is included in one component C of $\partial^s Y_{\Delta}$. This implies that S_1 and S_2 share a same end point. By the contradiction hypothesis, both curves of $\partial^u Y^g$ start from C in the opposite direction to Y_{Δ} . Thus the domain $\text{int } Y^g$, bounded by C and $\partial^u Y^g$, cannot intersect Y_{Δ} . A contradiction. \square

3.3 Generic points are eventually regular

We are now ready to split the support of invariant measures.

Proposition 3.11. *For every invariant measure ν with support off $\{A, A'\}$, ν -almost every point in Y_e is eventually $\sqrt{\Xi}$ -regular or satisfies $a_i(z) = \square$ for all i .*

Proof. We will show below that ν almost every point has a Lyapunov exponent positive:

Lemma 3.12. *There is no measure μ with both Lyapunov exponents negative.*

In §8, we will finish this proof by showing that if a point $z \in Y_e \setminus W^s(A)$ has a positive Lyapunov exponent and there is i such that $a_i(z) \neq \square$, then it is eventually $\sqrt{\Xi}$ -regular. \square

Proof of Lemma 3.12. By Lemma 3.9.1, the orbit of the eventually regular set supports only hyperbolic measures.

Thus a measure μ with both Lyapunov exponents negative must be supported by the orbit of the infinitely irregular points set.

By [KH95], corollary S.5.2, p.694, the support of μ contains a periodic attractive orbit $(p_i)_i$. Therefore $(p_i)_i$ is infinitely irregular. This implies that for an arbitrarily large N , there exists $\Delta = \square(c_N - c_{N+1})$ of depth N and i such that p_i belongs to Y_Δ . For N large, inequality $(\mathcal{CE}^{n_{c_i}+M+1})$ of Proposition 3.1 contradicts the attraction of the orbit. \square

4 Semi-conjugacy of regular points with a Markov shift

4.1 Symbolic sets

Let $\tilde{\mathcal{R}}$ be the set of Ξ -regular points in $Y_e \setminus W^s(A)$. Let \tilde{R} be the set of Ξ -regular infinite sequences. We notice that $\tilde{R} := \{\underline{a}(z) = (a_i(z))_i : z \in \tilde{\mathcal{R}}\}$. Let us define several subsets of

$$\mathfrak{A}^{(\mathbb{N})} := \cup_{m \geq 1} \{(a_i)_{i=1}^m \in \mathfrak{A}^m\}.$$

Let \mathfrak{R} be the set of finite Ξ -regular sequences:

$$\mathfrak{R} := \{\underline{a} \in \mathfrak{A}^{(\mathbb{N})} : \underline{a} \text{ is } \Xi\text{-regular}\}.$$

The set \mathfrak{R} is “almost” stable by concatenation:

Proposition 4.1. *Let $\underline{a}, \underline{a}' \in \mathfrak{R}$ be such that $f^{n_{\underline{a}}}(Y_{\underline{a}})$ intersects the interior of $Y_{\underline{a}'}$. Then $\underline{a} \cdot \underline{a}'$ belongs to \mathfrak{R} .*

Proof. We only need to prove that $\underline{a} \cdot \underline{a}'$ is suitable from $S^\#$. We proceed by induction on i with $\underline{a}' = a'_0 \cdots a'_i$. Let us suppose that $\underline{a} \cdot a'_0 \cdots a'_i$ is suitable from $S^\#$ and let $a'_{i+1} \in \mathfrak{P}(\# \cdot a'_0 \cdots a'_i)$ be such that $n_{a'_{i+1}} \leq M + \Xi n_{a'_0 \cdots a'_i}$. By Proposition 2.13, the curves $S^{\# \cdot \underline{a} \cdot a'_0 \cdots a'_i}$ and $S^{\# \cdot a'_0 \cdots a'_i}$ are $b^{n_{a'_0 \cdots a'_i}/4} \cdot C^1$ close. Thus by definition of the domain $D(a_{i+1})$, since $\Xi n_{a_1 \cdots a_{i+1}} + M \geq n_{a_{i+1}}$ and since $S^{\# \cdot \underline{a} \cdot a'_0 \cdots a'_i}$ must belong to the domain of a parabolic piece of same depth as a_{i+1} , the latter curve belongs to $D(a_{i+1})$, as $S^{\# \cdot a'_0 \cdots a'_i}$ does. This argument was generalized and proved in more details in Lemma 6.1 of [Ber11].

If $f^{n_{\underline{a}}}(Y_{\underline{a}})$ intersects $Y_{a'_0 \cdots a'_{i+1}} \setminus W^s(A)$ then $f^{n_{\underline{a} \cdot a'_0 \cdots a'_i}}(Y_{\underline{a} \cdot a'_0 \cdots a'_i})$ intersects $Y_{a'_{i+1}} \setminus W^s(A)$. Then we proceed as in the proof of Proposition 3.7 to show that $f^{n_{\underline{a} \cdot a'_0 \cdots a'_i}}(S^{\# \cdot \underline{a} \cdot a'_0 \cdots a'_i})$ intersects the interior of $S^{\# \cdot \underline{a} \cdot a'_0 \cdots a'_i}$. This implies that $\underline{a} \cdot a'_0 \cdots a'_{i+1}$ belongs to \tilde{R} . \square

Let $\tilde{\sigma}$ denote the shift dynamics of $\mathfrak{A}^{(\mathbb{N})}$; we extend it to the finite words set $\mathfrak{A}^{(\mathbb{N})}$:

$$\tilde{\sigma} : a_1 \cdots a_n \in \mathfrak{A}^{(\mathbb{N})} \mapsto a_2 \cdots a_n \in \mathfrak{A}^{(\mathbb{N})}.$$

It will be convenient to split the words in \tilde{R} into a concatenation of “minimal” words in \mathfrak{R} . This is why we introduce the subset \mathfrak{B} formed by words $g = g_1 \cdots g_j \in \mathfrak{R}$ for which there exists $\underline{a} = (a_i)_{i \geq 1} \in \tilde{R}$ such that:

(\mathfrak{B}_1) the concatenation $g \cdot \underline{a}$ belongs to \tilde{R} ,

(\mathfrak{B}_2) the ∞ -word $\tilde{\sigma}^n(g \cdot \underline{a})$ does not belong to \tilde{R} for $n \in (0, j)$.

The following is a simpler and an equivalent definition of \mathfrak{B} :

Proposition 4.2. *A word $g \in \mathfrak{R}$ belongs to \mathfrak{B} iff the following conditions hold:*

(\mathfrak{B}'_1) *there exists $s \in \mathfrak{Y}_0$ such that $g \cdot s$ is suitable from \mathfrak{t} ,*

(\mathfrak{B}'_2) *for every $n \geq 1$, $\tilde{\sigma}^n(g)$ is not Ξ -regular.*

Proof. Let $g = g_1 \cdots g_j \in \mathfrak{A}^{(\mathbb{N})}$ be satisfying (\mathfrak{B}'_1) and (\mathfrak{B}'_2) with $s \in \mathfrak{Y}_0$. We remark that the suitable sequence $g \cdot s$ from $S^\mathfrak{t}$ is complete. By (SR_3) it is the \mathfrak{A} -spelling of a product of puzzle pieces in \mathcal{Y} . Thus $S^{\mathfrak{t} \cdot g \cdot s}$ is equal to $f^{n_{g \cdot s}}(S_{g \cdot s}^\mathfrak{t})$ and so $g \cdot s \cdot s_+ \cdots s_+ \cdots$ is suitable from $S^\mathfrak{t}$. As g belongs to \mathfrak{R} , it is now clear that this sequence is Ξ -regular: this is (\mathfrak{B}_1). By (B'_2), it is effortless to see that $\tilde{\sigma}^i(g) \cdot \underline{a}$ does not belong to \tilde{R} for every $i \leq j$: this is (\mathfrak{B}_2).

Let us show that (\mathfrak{B}_1) and (\mathfrak{B}_2) imply (\mathfrak{B}'_1) and (\mathfrak{B}'_2). For $g = g_1 \cdots g_j \in \mathfrak{R}$ and $\underline{a} = (a_i)_i \in \tilde{R}$ such that $g \cdot \underline{a}$ belongs to \tilde{R} , it holds that $g \cdot a_1$ is suitable from $S^\mathfrak{t}$. As regular sequences begin with symbols in \mathfrak{Y}_0 , the symbol a_1 is in \mathfrak{Y}_0 . This is (\mathfrak{B}'_1).

For $n < j$, let us suppose that the word $\tilde{\sigma}^n(g)$ belongs to \mathfrak{R} . As $g \cdot \underline{a}$ is suitable from $S^\mathfrak{t}$, the segment $S_{\tilde{\sigma}^n(g) \cdot a_1 \cdots a_i}^{\mathfrak{t} \cdot g_1 \cdots g_n}$ is non trivial for every $i \geq 1$. Thus $f^{n_{\tilde{\sigma}^n(g)}}(Y_{\tilde{\sigma}^n(g)})$ intersects the interior of $Y_{a_1 \cdots a_i}$ for every i . By Proposition 4.1, the word $\tilde{\sigma}^n(g) \cdot a_1 \cdots a_i$ belongs to \mathfrak{R} for every $i \geq 1$. This is a contradiction with (\mathfrak{B}_2). □

The set $\mathfrak{G} := \{e\} \cup \bigcup_{n \geq 0} \tilde{\sigma}^n(\mathfrak{R})$ is fundamental. One can show that \mathfrak{G} consists of the segments of the words in $\mathfrak{A}^{(\mathbb{N})}$ which are suitable from a curve S^t , for some $t \in T$.

We remark that:

$$\mathfrak{A} \subset \mathfrak{G} \supset \mathfrak{R} \supset \mathfrak{B}$$

are endowed with a pseudo-group structure “.” and their elements act on subspaces of flat stretched curves by graph transforms.

Let R be the subset of words in $\underline{a} \in \tilde{R}$ which come back infinitely many times to \tilde{R} :

$$R := \{\underline{a} \in \tilde{R}; \forall N \geq 0, \exists n \geq N, \tilde{\sigma}^n(\underline{a}) \in \tilde{R}\}.$$

Let \mathcal{R} be the corresponding subset of $\tilde{\mathcal{R}}$:

$$\mathcal{R} := \{z \in \tilde{\mathcal{R}} : \underline{a}(z) = (a_i(z))_i \in R\}.$$

For every $\underline{a} \in R$, let $N_{\underline{a}}$ be the first return time of \underline{a} in \tilde{R} by the shift map $\tilde{\sigma}$. We notice that $g_{\underline{a}} := a_1 \cdots a_{N_{\underline{a}}}$ belongs to \mathfrak{B} . The associated map is:

$$F := z \in \mathcal{R} \mapsto f^{N(z)} \in \mathcal{R} \text{ with } N(z) := N_{\underline{a}(z)}.$$

Let $\tilde{\mathcal{R}} := \bigcap_{n \geq 0} F^n(\mathcal{R})$. Moreover, by looking at the Lyapunov exponent we will show in §8:

Proposition 4.3. *For every f -invariant probability μ , the sets $\cup_{n \geq 0} f^n(\tilde{\mathcal{R}})$ and $\cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$ are equal μ -almost everywhere.*

We will see that the restriction of F to $\tilde{\mathcal{R}}$ is the first return map of f to $\tilde{\mathcal{R}}$ in Proposition 6.1.

4.2 Markov partition of \mathcal{R}

We recall that for $\underline{a} = (a_i)_i \in R$, we denote by $N_{\underline{a}}$ its first return in \tilde{R} by the shift map $\tilde{\sigma}$ and put $g_{\underline{a}} := a_0 \cdots a_{N_{\underline{a}}-1}$. We recall that $g_{\underline{a}}$ belongs to \mathfrak{B} . For every $g \in \mathfrak{B}$, let:

$$\mathcal{R}_g := \{x \in \mathcal{R} : g_{\underline{a}(x)} = g\}.$$

We remark that $\{\mathcal{R}_g; g \in \mathfrak{B}\}$ is a partition of \mathcal{R} . As a matter of fact, \mathcal{R}_g is not equal to $Y_g \cap \mathcal{R}$, for $\mathcal{R} \subset \cup_{\mathfrak{Y}_0} Y_s$ and \mathfrak{Y}_0 is a subset of \mathfrak{B} . The aim of this paragraph is to prove:

Proposition 4.4. *The partition $(\mathcal{R}_g)_{g \in \mathfrak{B}}$ is Markovian.*

To say that this partition is *Markovian* means that for every g, g' such that $f^{n_g}(\mathcal{R}_g) \cap \mathcal{R}_{g'} \neq \emptyset$, for every $x' \in \mathcal{R}_{g'}$ there exists $x \in \mathcal{R}_g$ for which $f^{n_g}(W_{\underline{a}(x)}^s) \subset W_{\underline{a}(x')}^s$.

Proposition 4.4 follows from the following:

Lemma 4.5. *Let $g, g' \in \mathfrak{B}$ be such that $f^{n_g}(\mathcal{R}_g)$ intersects $\mathcal{R}_{g'}$. For every $x' \in \mathcal{R}_{g'}$ there exists $x \in S^\#$ such that $\underline{a}(x) = g \cdot \underline{a}(x')$ and $f^{n_g}(x) \in W_{\underline{a}(x')}^s$.*

Indeed such a point x is regular by definition and $g \cdot \underline{a}(x)$ belongs to R since $\underline{a}(x)$ does.

Moreover, as the stable manifold $f^{n_g}(W_{\underline{a}(x)}^s)$ is included in the interior of Y_e and the end points of $W_{\underline{a}(x')}^s$ are in $\partial^u Y_e$, it comes that $f^{n_g}(W_{\underline{a}(x)}^s) \subset W_{\underline{a}(x')}^s$.

Proof of Lemma 4.5. Let $s \in \mathfrak{Y}_0$ be the first letter of g' . As $Y_{g'} \subset Y_s$, the square $f^{n_g}(Y_g)$ intersects Y_s . From the geometries of Y_s and $f^{n_g}(Y_g)$ (see box definition and Lemma 3.9), this implies that $f^{n_g}(S_g^\#)$ intersects Y_s . As $S^{\# \cdot g}$ belongs to the domain $D(s)$ (as every flat stretched curve), the word $g \cdot s$ is suitable from $S^\#$. By (SR_3) , the pair $(g \cdot s)(S^\#)$ is a puzzle piece of $S^\#$. In particular $f^{n_g}(S_g^\#)$ stretches across Y_s . Thus for every $x' \in \mathcal{R}_{g'} \subset Y_s$, there exists $x \in S_g^\#$ such that $f^{n_g}(x) \in W_{\underline{a}(x')}^s$.

We remark that $\underline{a}(x) = g \cdot \underline{a}(x')$. Indeed, as g and $\underline{a}(x')$ are Ξ -regular and $f^{n_g}(Y_g) \cap W_{\underline{a}(x')}^s \neq \emptyset$, so we can apply Proposition 4.1. \square

4.3 Markovian model of the $\tilde{\mathcal{R}}$ -orbit

We are going to conjugate the dynamic of $\cup_{n \geq 0} f^n(\tilde{\mathcal{R}})$ -orbit with a Markov countable, mixing shift without the stable set of a 2-periodic point.

Let us recall that a countable shift is defined by a graph G with vertices V and arrows $\Pi \subset V^2$. Let Ω_G be the set of infinite two-sided sequences $(v_n)_n \in V^{\mathbb{Z}}$ such that $(g_n, g_{n+1}) \in \Pi$ for every n . The shift map of Ω_G is denoted by σ .

The idea is to use the \mathfrak{B} alphabet to define V . As we want to conjugate f and not F , we should use a graph Γ with the following vertices:

$$\Upsilon := \{(g, i) : g \in \mathfrak{B}, 0 \leq i \leq n_g - 1\},$$

and arrows:

$$\bigcup_{g \in \mathfrak{B}} \{[(g, i), (g, i + 1)] : 0 \leq i \leq n_g - 2\} \sqcup \{[(g, n_g - 1), (g', 0)] : g' \in \mathfrak{B}, f^{n_g}(\mathcal{R}_g) \cap \mathcal{R}_{g'} \neq \emptyset\}.$$

Unfortunately such a graph does not seem to be strongly positive recurrent. We shall emphasis on the fact that there is a “first return” to $\tilde{\mathcal{R}}$. A first idea would be to replace the vertex $(g, 0)$ by e . However it is (in general) not true that for every $g, g' \in \mathfrak{B}$ there exists a point $x \in \tilde{\mathcal{R}}$ such that $\underline{a}(x)$ starts by the \mathfrak{A} spelling of $g \cdot g'$. Instead, we will notice in corollary 6.6 that for $s, s' \in \mathfrak{Y}_0$ and $s \cdot g' \in \mathfrak{B}$, the word $s' \cdot g'$ belongs to \mathfrak{B} as well. This leads us to work with the oriented countable graph G formed by the following vertices V and arrows Π :

$$V := \bigcup_{g=s \cdot g' \in \mathfrak{B}} \{(s, i) : 0 \leq i \leq n_s - 2\} \cup \{e\} \sqcup \{(g', i) : 0 \leq i \leq n_{g'} - 1\}$$

$$\begin{aligned} \Pi := & \bigcup_{g=s \cdot g' \in \mathfrak{B}} \{[(s, i), (s, i + 1)] : 0 \leq i \leq n_s - 3\} \sqcup \{[(s, n_s - 2), e]\} \sqcup \{[e, (g', 0)]\} \\ & \sqcup \{[(g', i), (g', i + 1)] : 0 \leq i \leq n_{g'} - 2\} \sqcup \{[(g', n_{g'} - 1), (s', 0)] : s' \in \mathfrak{Y}_0, f^{n_g}(\mathcal{R}_g) \cap Y_{s'} \neq \emptyset\}. \end{aligned}$$

Note that there is a canonical map γ from Υ to V : for $g = s' \cdot g' \in \mathfrak{B}$, put:

$$\gamma(g, i) = \begin{cases} (s', i) & \text{if } i \leq n_{s'} - 2, \\ e & \text{if } i = n_{s'} - 1, \\ (g', i - n_{s'}) & \text{if } i \geq n_{s'}. \end{cases}$$

The map γ is not a bijection. It is surjective but not injective.

Lemma 4.6. *For any $(v, v') \in \Upsilon^2$, the pair $[v, v']$ is an arrow of Γ iff $[\gamma(v), \gamma(v')]$ is an arrow of G .*

Proof. It is clear that γ sends arrows of Γ to arrows of G . The other direction it given by Corollary 6.6 which shows that if $s \cdot g'$ belongs to \mathfrak{B} then $s' \cdot g'$ belongs to \mathfrak{B} for all $s, s' \in \mathfrak{Y}_0$. \square

Therefore, the map γ induces a canonical semi-conjugacy I between the shift of the space $\Omega_\Gamma \subset \Upsilon^\mathbb{Z}$ of paths in Γ and the shift of the space $\Omega_G \subset V^\mathbb{Z}$ of paths in G .

Corollary 4.7. *The semi-conjugacy $I : \Omega_\Gamma \rightarrow \Omega_G$ is a bijection and so a conjugacy.*

Proof. Let $(v_i)_{i \in \mathbb{Z}}$ and $(w_i)_{i \in \mathbb{Z}} \in \Omega_\Gamma$ be sent to a same point by I . Using the semi-conjugacy, we can suppose that v_0 is of the form $(g, 0)$. Observe that $v_i = (g, i)$, for $0 \leq i < n_g$. Put $g = s \cdot g'$ with $s \in \mathfrak{Y}_0$. As $\gamma(w_i) = \gamma(v_i) = (s, i)$ for $i \leq n_s - 2$ and $\gamma(w_i) = \gamma(v_i) = (g', i - n_s)$ for $i \geq n_s$, it comes that $w_i = v_i$ for $0 \leq i < n_g$. Using again the semi-conjugacy, this implies even that $\underline{w} = \underline{v}$. \square

Let $\tilde{A} := (a_i)_i \in V^{\mathbb{Z}}$, where $a_{2i} = (s_-, 0)$ and $a_{2i+1} = e$ for every i . This is a 2-periodic fixed point which corresponds to the fixed point A endowed with an orientation.

Let $\Omega'_G := \Omega_G \setminus (W^s(\tilde{A}) \sqcup W^s(\sigma(\tilde{A})))$, where $W^s(\tilde{A})$ and $W^s(\sigma(\tilde{A}))$ are the stable sets for the dynamics $\sigma : \Omega_G \rightarrow \Omega_G$ of the points \tilde{A} and $\sigma(\tilde{A})$.

Let $\Omega'_\Gamma := \Omega_\Gamma \setminus (W^s(s_-, 0) \sqcup W^s(s_-, 1))$.

We endow Ω_G with the following distance:

$$d(\underline{v}, \underline{v}') = 2^{-\nu(\underline{v}, \underline{v}')} , \quad \nu(\underline{v}, \underline{v}') := \inf\{|k| : v_k \neq v'_k\}$$

In the next subsections, we will show the following:

Proposition 4.8. *There exists a bijection $i : \Omega'_G \rightarrow \mathcal{O}(\check{\mathcal{R}})$, with $\mathcal{O}(\check{\mathcal{R}}) := \bigcup_{n \geq 0} f^n(\check{\mathcal{R}})$, which is Hölder continuous and such that the following diagram commutes:*

$$\begin{array}{ccccc} \Omega'_\Gamma & \xrightarrow{I} & \Omega'_G & \xrightarrow{i} & \mathcal{O}(\check{\mathcal{R}}) \\ \sigma \downarrow & & \sigma \downarrow & & f \downarrow \\ \Omega'_\Gamma & \xrightarrow{I} & \Omega'_G & \xrightarrow{i} & \mathcal{O}(\check{\mathcal{R}}) \end{array}$$

and is such that for $g \in \mathfrak{B}$ and $x \in \check{\mathcal{R}} \cap \mathcal{R}_g$, the 0-coordinate of $I^{-1} \circ i^{-1}(x)$ is $(g, 0)$.

The shift σ of Ω_G enjoys of many nice properties.

Proposition 4.9. *The shift σ is topologically mixing.*

Proof. For all $\underline{v} = (v_i)_i \in \Omega_G$ and $n \geq 0$, let:

$$C_{n, \underline{v}} := \{(v'_i)_i \in \Omega_G : v_i = v'_i, \forall |i| \leq n\}.$$

The family $(C_{n, \underline{v}})_{n, \underline{v}}$ is a base of neighborhoods of Ω_G . For all $\underline{v}, \underline{v}' \in \Omega_G$, all $n, n' \in \mathbb{N}$, we want to show that there exists $N \geq 0$ such that for every $k \geq N$, $\sigma^k(C_{n, \underline{v}})$ intersects $C_{n', \underline{v}'}$. By taking n and n' larger, without loss of generality, we can assume that $v_n = e$ and $v'_{-n'} = e$.

We remark that for every $l \geq 2$, there exists an admissible loop with base point e of length l . Indeed, by \hat{G}_7 , there exist a loop of length 2 which is $[e, (s_-, 0), e]$ and a loop of length 3 of the form $[e, (s, 0), (s, 1), e]$, with $s \in \mathfrak{Y}_0$.

Thus for every $k \geq N := n + n' + 2$, $\sigma^k(C_{n, \underline{v}})$ intersects $C_{n', \underline{v}'}$. □

To study the ergodic properties of σ , we regard the number Z_n^* of loops in the graph G passing by e only at the begin and the end.

Let R_* be the convergence radius of the series $\sum_n Z_n^* X^n$. Remember that $\epsilon = 1/\sqrt{M}$.

Proposition 4.10. *The convergence radius R_* is greater than $e^{-2\epsilon}$.*

We recall since the entropy of the shift is at least close to $\log 2$, the two latter propositions imply:

Corollary 4.11. *The shift σ is strongly positive recurrent.*

Proof of Proposition 4.10. To bound R_* from below, we are going to show that:

$$(4.1) \quad Z_n^* \leq 2e^{2\epsilon n}, \quad \forall n \geq 0.$$

We notice that:

$$Z_n^* = \text{Card} \{g \in \mathfrak{B} : n_g = n\}.$$

First, let us prove that:

$$(4.2) \quad P_m := \sup_{t \in T} \text{Card} \{a_1 \cdots a_j \in \mathfrak{Y}_0 \times (\mathfrak{A} \setminus \mathfrak{Y}_0)^{(\mathbb{N})} : t \cdot a_1 \cdots a_j \in T^\square \sqcup T, n_{a_1 \cdots a_j} = m\} \leq 2e^{\epsilon m}$$

It is the cardinality of the subset of suitable sequences from S^t , beginning by a simple symbol and then equal only to parabolic symbols, such that the sum of their orders is equal to m . The first symbol has an order at most M , and the other symbols have an order at least $M + 1$. Also, given $(a_i)_{i=1}^j$ suitable from $S^\#$, for each $k \geq 2$, there are at most two symbols a_{j+1} in \mathfrak{A} of order k such that $(a_i)_{i=1}^{j+1}$ is suitable from $S^\#$ (this is clear when a_{j+1} is parabolic, when it is simple it follows from \hat{G}_7).

Consequently, it holds $P_n = 2$ for $2 \leq n \leq M$ and for $n \geq M + 1$:

$$P_n \leq 2 \sum_{k=M+1}^n P_{n-k} = 2 \sum_{k=2}^{n-M-1} P_k.$$

Thus if $M + 1 \leq n \leq 2M + 1$, by \hat{G}_7 , these two inequalities imply $P_n \leq 4(n - M - 2) \leq 2e^{\epsilon n}$. If $n \geq 2M + 2$, the induction gives:

$$P_n \leq 2 \sum_{k=2}^{n-M-1} 2e^{\epsilon k} \leq 4e^{2\epsilon} \frac{1 - e^{\epsilon(n-M-1)}}{1 - e^\epsilon} \leq 4 \frac{e^{\epsilon(n-M+1)}}{\epsilon}$$

This proves that P_n is less than $2e^{\epsilon n}$, since $n \geq 2M + 1$ implies that $e^{\epsilon n}$ is much larger than $M 4e^{-\epsilon M}/\epsilon$ is very small, by Remark 2.15.

Let $g = (a_i)_{i=1}^N \in \mathfrak{B}$ be such that $n_g = n$. There are two possibilities.

Either a_i belongs to \mathfrak{Y}_0 only when $i = 1$, and the cardinality of such a possibility is at most P_n .

Either there exists $i_0 > 1$ maximal such that a_{i_0} belongs to \mathfrak{Y}_0 . Put $g' := a_1 \cdots a_{i_0-1}$ and $\underline{a} := a_{i_0} \cdots a_N$. By definition of \mathfrak{B} , \underline{a} does not belong to \mathfrak{Y}_0 . The cardinality of such g' is given by $Z_{n_{g'}}^*$, while the cardinality of such \underline{a} is given by $P_{n_{\underline{a}}}$. As $\underline{a} \notin \mathfrak{Y}_0$, $n_{\underline{a}} > M$. We recall that $Z_n^* = 2$ for $n \leq M$. Let $n \geq M + 1$.

$$Z_n^* - P_n \leq \sum_{k=0}^{n-M-1} Z_k^* \cdot P_{n-k} \leq \sum_{k=0}^{n-M-1} 4e^{2\epsilon k} e^{\epsilon(n-k)} \leq 4 \frac{e^{\epsilon(n-M)+\epsilon n}}{\epsilon}.$$

As $e^{-\epsilon M}/\epsilon$ is very small, $Z_n^* - P_n$ is very small with respect to $2e^{2\epsilon n}$, for $n \geq M + 1$. \square

4.4 Construction of i

Now, we are ready to define the conjugacy:

Proposition 4.12. *There exists a map*

$$i : \Omega'_G \rightarrow \bigcup_{n \geq 0} f^n(\tilde{\mathcal{R}})$$

for which the diagram of Proposition 4.8 commutes and satisfies the following: for all $\underline{w} = (w_k)_k \in \Omega'_G$ and $g \in \mathfrak{B}$ such that the zero coordinate of $I^{-1}(\underline{w})$ is $(g, 0)$, the point $i(\underline{w})$ belongs to \mathcal{R}_g .

Proof. Let $(g_i)_{i \in \mathbb{Z}} \in \mathfrak{B}^{\mathbb{Z}}$ be such that $f^{n_{g_i}}(\mathcal{R}_{g_i}) \cap \mathcal{R}_{g_{i+1}} \neq \emptyset$ for every $i \in \mathbb{Z}$, and $(g_i)_i$ is not eventually equal to s_- . By Lemma 4.5 and an induction on N , the curve S^\sharp intersects $\cap_{j=N'}^N f^{-n_{g_{N'}} \cdots g_0 \cdots g_j}(\mathcal{R}_{g_{j+1}})$ for every $N' \leq 0 \leq N$. Moreover for x in this intersection, $\underline{a}(x)$ begin with $g_{N'} \cdots g_N$ and so is in \mathfrak{R} . Therefore, $g_{N'} \cdots g_N \cdots$ is in \tilde{R} (and even in R). This enables us to regard the curve $W_{g_{N'} \cdots g_N}^s$ given by Corollary 3.5. By Lemma 3.9.2, the length of $f^{n_{g_{N'}} \cdots g_1}(W_{g_{N'} \cdots g_N}^s)$ is smaller than $\theta^{n_{g_{N'}} \cdots g_1}$, for every $N' \leq 0$. This decreasing sequence of curves converges to a single point x_0 as $N' \rightarrow -\infty$.

Let $\underline{w} = (w_k)_k \in \Omega_G$ be the canonical sequence induced by $(g_i)_i$ such that the zero coordinate of $I^{-1}(\underline{w})$ is $(g_0, 0)$. Put $i(\underline{w}) = x_0$ and $i \circ \sigma^n(\underline{w}) = f^n(x)$. \square

5 Geometrical and dynamical properties of the Hénon attractor

Since the Hénon attractor support an SRB, the unstable Hausdorff dimension of the Hénon attractor is one. The stable Hausdorff dimension of the Hénon attractor is more delicate to compute.

5.1 Stable Hausdorff dimension of the Hénon attractor

The stable Hausdorff dimension can be defined for points which admit a Pesin stable manifold, and so for points which are generic for some measure. These points are all eventually regular by Proposition 3.11. This means that they are sent by some iterates into $\tilde{\mathcal{R}}$. One can show then that every ergodic measure has its support included in $\overline{\lim} \tilde{\mathcal{R}} := \cap_{N \geq 0} \cup_{n \geq N} f^n(\tilde{\mathcal{R}})$.

We recall that every point $x \in \overline{\lim} \tilde{\mathcal{R}}$ belongs to an intersection of the form $\cap_{i \geq 0} f^{n_i}(W_{\underline{a}_i}^s)$ with $\underline{a}_i \in \tilde{R}$ and $(n_i)_i$ an increasing sequence. This implies that x belongs to the image by f^{n_j} of $W_{\underline{a}_j}^s \cap \cap_{i \geq j} f^{n_i - n_j}(W_{\underline{a}_i}^s)$, for n_j arbitrarily large. Thus x belongs to $f^{n_j}(W_{\underline{a}_j}^s)$.

Proposition 5.1. *For every $\underline{a} \in \tilde{R}$, the set $W_{\underline{a}}^s \cap \overline{\lim} \tilde{\mathcal{R}}$ has a Hausdorff dimension small for b small.*

Proof. The set $W_{\underline{a}}^s \cap \overline{\lim} \tilde{\mathcal{R}}$ is included in $\cup_{n \geq N} \cup_{\underline{a}' \in \tilde{R}} f^n(W_{\underline{a}'}^s) \cap W_{\underline{a}}^s$ for every $N \geq 0$. For every $\underline{a}' \in \tilde{R}$ and $n \geq N$, let $\underline{a}'_n \in \mathfrak{R}$ be shortest word with order $n_{\underline{a}'_n} \geq n$ of \underline{a}'_n and with the property that $\underline{a}' = \underline{a}'_n \cdot \underline{b}$ for $\underline{b} \in \mathfrak{A}^{\mathbb{N}}$. Remark that $n_{\underline{a}'_n}$ is less than $\Xi n + M$.

The length of $f^n(Y_{\underline{a}'_n}) \cap W_{\underline{a}}^s$ is smaller than θ^n by Lemma 3.9. Also the cardinality of $\{\underline{a}'_n : \underline{a}' \in \tilde{R}\}$ is less than $2^{\Xi n + M}$ by the following Lemma shown below:

Lemma 5.2. *For every $N \geq 0$, the following cardinality is less than 2^N :*

$$\#_N := \sup_{t \in T \sqcup T^\square} \text{Card}\{\underline{a}' \in \mathfrak{A}^{(\mathbb{N})} : n_{\underline{a}'} = N, t \cdot \underline{a}' \in T \sqcup T^\square\}.$$

Thus the Hausdorff dimension of $W_{\underline{a}}^s \cap \overline{\lim} \tilde{\mathcal{R}}$ is smaller than $\Xi \log 2 / |\log \theta|$ since it is the supremum of s among which the series $\sum_{n \geq N} \theta^{sn} 2^{\Xi n + M}$ converges. \square

Remark 5.3. The estimate proved in the above lemma is not optimal at all. It seems to me possible to prove that the Hausdorff dimension is smaller than a constant times $\theta = 1/|\log b|$

Proof of Lemma 5.2. We proceed by induction on N . For $N \leq 2$, it follows from \hat{G}_7 that $\#_2 = 2$ and $\#_1 = 0$.

In general, for each $t \in T \sqcup T^\square$ and $n \geq 0$, there at most two symbols in \mathfrak{A} suitable from S^t and with order n (by \hat{G}_7 for the simple symbol, i.e. $n \leq M$). Thus for $N > 2$, by induction:

$$\#_N \leq 2 + \sum_{n=2}^{N-2} 2\#_{N-n} \leq 2 + \sum_{n=2}^{N-2} 2^{n+1} \leq 2^N$$

\square

5.2 Observations on the geometry of $\tilde{\mathcal{R}}$

Let us give briefly the flavor of the geometry of $\tilde{\mathcal{R}}$.

We already saw that every point $x \in \tilde{\mathcal{R}}$ enjoys of a long stable manifold W_x^s . The length of this curve is at least 2θ . A same proof as for Proposition 3.9 of [Ber11] shows that W_x^s is a Lipschitz function of $x \in \tilde{\mathcal{R}}$ in the space of C^1 -curves.

For every $s \in \mathfrak{Y}_0$, a point $x \in \tilde{\mathcal{R}} \cap Y_s$ enjoys also of a long unstable manifold: the point x belongs to the “non-artificial slice” of a unique curve S^t , $t \in T \sqcup T^\square$, which stretches across Y_s . This “non-artificial slice” is a segment W_x^u containing x called (long) local unstable manifold of x . From Proposition 5.17 of [Ber11], the C^1 -curve W_x^u is a Holder function of $i(x)$, for a suitable metric on Ω_G .

Remark 5.4. With respect to these long stable and unstable manifolds, the set $\tilde{\mathcal{R}}$ has a product structure: the intersection of $W_x^u \cap W_y^s$ is a single point of $\tilde{\mathcal{R}}$ for all $x, y \in \tilde{\mathcal{R}} \cap Y_s$, $s \in \mathfrak{Y}_0$.

The partition $(\mathcal{R}_g)_g$ of \mathcal{R} is actually wild, although W^s -saturated. For every $g \in \mathfrak{B}$, any two connected components of \mathcal{R}_g are separated by infinitely many others $\mathcal{R}_{g'}$.

5.3 Dynamics on K_\square

By Proposition 3.11, every generic point $x \in Y_e \setminus W^s(A)$ is either eventually regular or lies in the following invariant compact set:

$$K_\square := \bigcap_{N \geq 0} \bigcup_{n \geq N} f^n(\{\underline{a}(x) = \square \cdots \square \cdots : x \in Y_e \setminus W^s(A)\}).$$

This set is actually pretty simple. Let $\Delta_{\pm} := \square_{\pm}(e - c_1(\#))$ be the two parabolic symbols of depth $M + 1$ associated to $S^{\#}$. We remark that

$$\{\underline{a}(x) = \square \cdots \square \cdots : x \in Y_e\} = \bigcap_{n \geq 0} f^{-n(M+1)}(Y_{\Delta_-} \cup Y_{\Delta_+}).$$

One can show that for our parameters, $f^{M+1}(\partial^s Y_{\square})$ is included in the component of $\partial^s Y_e$ containing A (see fig 2). Thus if $c_1(\#) = s_-$, then the boxes Y_{Δ_-} and Y_{Δ_+} are empty and so K_{\square} is empty. If the simple symbol $c_1(\#)$ has its box $Y_{c_1(\#)}$ at the left of Y_{\square} , then $f^{M+1}(Y_{\Delta_-} \cup Y_{\Delta_+})$ does not intersect $Y_{\Delta_-} \cup Y_{\Delta_+}$, and K_{\square} is empty again. Otherwise, $f^{M+1}|K_{\square}$ is a horseshoe conjugated to the shift of $\{0, 1\}^{\mathbb{Z}}$. This proves:

Proposition 5.5. *If K_{\square} is not empty, the topological entropy of $f|K_{\square}$ is $\log 2/(M + 1)$.*

As M is large, the latter entropy is small and so K_{\square} does not intersect the support of a maximal entropy probability.

Therefore a generic point of maximal entropy measure is eventually $\sqrt{\Xi}$ -regular. As we already showed the uniqueness of the maximal entropy measure supported by the orbit of $\tilde{\mathcal{R}}$, it remains only to show that the orbit of $\mathcal{E}_0 := \tilde{\mathcal{R}} \setminus \mathcal{R}$ does not support maximal entropy probability. Actually the invariant probability supported by the orbit of \mathcal{E}_0 are supported by $\mathcal{E} := \bigcap_{N \geq 0} \bigcup_{n \geq N} f^n(\mathcal{E}_0)$. We remark that \mathcal{E} is f -invariant. The two following sections are devoted to the study of the invariant probabilities supported by \mathcal{E} .

5.4 Invariant probabilities in \mathcal{E}

Whereas alphabet \mathfrak{B} was suitable to study \mathcal{R} , we shall use the following alphabet² to study \mathcal{E} :

$$\mathfrak{M} := \bigcup_m \{a_1 \cdots a_m \in \mathfrak{G} : n_{a_m} > M + \Xi \sum_{k < m} n_{a_k}\}.$$

It is indeed useful to encode the set $E_0 := \tilde{R} \setminus R$ which satisfies $\mathcal{E}_0 = \bigcup_{\underline{a} \in E_0} W_{\underline{a}}^s$.

Proposition 5.6. *For every $\underline{a} \in E_0$, there exist $a' \in \mathfrak{R}$ and $(m_i)_{i \geq 0} \in \mathfrak{M}^{\mathbb{N}}$ such that:*

$$\underline{a} = a' \cdot m_1 \cdots m_n \cdots$$

Proof. If \underline{a} does not come back infinitely often to \tilde{R} , then for every n large, let $N \geq 0$ be such that for every $n \geq N$, $\tilde{\sigma}^n(\underline{a}(z))$ does not belong to \tilde{R} . Let \underline{a}' be the word formed by the $N - 1$ first symbols of \underline{a} . Let \underline{a}_0 be such that $\underline{a} = \underline{a}' \cdot \underline{a}_0$. We remark that $\tilde{\sigma}^n(\underline{a}_0) \notin \tilde{R}$ for every $n \geq 0$. Thus there exists $m_1 \in \mathfrak{M}$ such that $\underline{a}_0 := m_1 \cdot \underline{a}_1$, for a certain $\underline{a}_1 \notin \tilde{R}$. Moreover $\tilde{\sigma}^n(\underline{a}_1) \notin \tilde{R}$, for every $n \geq 0$, and so we can write $\underline{a}_1 := m_2 \cdot \underline{a}_2$ with $m_2 \in \mathfrak{M}$ and $\underline{a}_2 \notin \tilde{R}$. And so on the proposition comes by induction. \square

²See the index for the definition of \mathfrak{G} .

Let \mathcal{E}'_0 be the subset of \mathcal{E}_0 of $\sqrt{\Xi}$ -regular points x :

$$n_{a_i(x)} \leq M + \sqrt{\Xi} \sum_{j < i} n_{a_j(x)} \text{ for every } i > 0.$$

Similarly we define $\mathcal{E}' := \cap_{n_0 \geq 0} \cup_{n \geq n_0} f^n(\mathcal{E}'_0)$. By Proposition 3.11, any ergodic probability ν supported by \mathcal{E} is actually supported by \mathcal{E}' . Consequently $\nu(\mathcal{E}'_0) > 0$. Let:

$$E'_{0,N} := \{\underline{a} \in E_0 : \underline{a} \text{ is } \sqrt{\Xi}\text{-regular and } \underline{a} = \underline{a}' \cdot b_1 \cdots b_n \cdots \text{ with } \underline{a}' \in \mathfrak{R}, b_i \in \mathfrak{M}, n_{\underline{a}'} \leq N\},$$

$$\mathcal{E}'_{0,N} := \{x \in \mathcal{E}'_0 : \underline{a}(x) \in E'_{0,N}\}.$$

We remark that $\mathcal{E}'_0 = \cup_{N \geq 0} \mathcal{E}'_{0,N}$. As the latter union is increasing, there exists N such that $\nu(\mathcal{E}'_{0,N}) > 0$ and so $\nu(\mathcal{E}'_N) > 0$, with:

$$\mathcal{E}'_N := \bigcap_{n_0 \geq 0} \bigcup_{n \geq n_0} f^n(\mathcal{E}'_{0,N}).$$

By ergodicity, the support of ν is actually contained in \mathcal{E}'_N : $\nu(\mathcal{E}'_N) = 1$.

In the next section the following is shown:

Proposition 5.7. *For every $N \geq 0$, the set \mathcal{E}'_N has Hausdorff dimension smaller than $3/\sqrt{M}$.*

From this we can deduce:

Corollary 5.8. *The set \mathcal{E} does not support a measure of high entropy.*

Proof. Young's entropy formula [You82] for any ergodic invariant probability is the following:

$$d_\nu = h_\nu \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right)$$

with h_ν the entropy of ν , d_ν the infimum of the Hausdorff dimension of a set of full measure, and λ_1, λ_2 the Lyapunov exponent of ν . The support of any ergodic probability ν included in \mathcal{E} is actually included in \mathcal{E}'_N for a certain $N \geq 0$, thus the dimension d_ν is small by Proposition 5.7. As the norm of the differential is less than e^{c^+} and the jacobian very small, it comes that h_ν is smaller than $2c^+d_\nu$ and so is smaller than $6c^+/\sqrt{M} \ll 1$. \square

5.5 Hausdorff dimension of \mathcal{E}

To show that the Hausdorff dimension of \mathcal{E}'_N is small, we work with a family of nice coverings of $K_* := S^\# \cap \mathcal{E}_0$.

Let $|Y_e|$ be the *width* of Y_e that is the maximal length of a flat stretched curve. For every curve \mathcal{C} , let us denote by $|\mathcal{C}|$ its length.

Lemma 5.9. *For every N , there exists $C_N \subset \mathfrak{R}$ such that:*

- (i) $\{S^\#_{\underline{a}} : \underline{a} \in C_N\}$ covers K_* ,

(ii) for every $\underline{a} \in C_N$, $n_{\underline{a}} \geq N$ and $|S_{\underline{a}}^{\#}| \leq |Y_e|e^{-n_{\underline{a}}c/3} \leq |Y_e|e^{-cN/3}$,

(iii) $\sum_{\underline{a} \in C_N} e^{-sn_{\underline{a}}c/3} < \lambda^N$, with $\lambda := e^{-M^{1/4}}$ and $s := 1/\sqrt{M}$.

An immediate consequence is the following:

Corollary 5.10. *The set K_* has Hausdorff dimension smaller than $s = 1/\sqrt{M}$, and so is small for M large.*

Actually the same estimate holds for every $S^t \cap \mathcal{E}_0$, $t \in T \sqcup T^\square$.

Proof of Lemma 5.9. For every N , let:

$$C'_N := \{\underline{a}' \cdot b_1 \cdots b_N \in \mathfrak{R} : b_i \in \mathfrak{M}, \underline{a}' \in \mathfrak{R}, n_{\underline{a}'} \leq N\}.$$

We remark that $C_N := \cup_{N' \geq N} C'_{N'}$ satisfies (i) by Proposition 5.6. Property (ii) holds by Lemma 3.9.

Let us show (iii). For $s > 0$, put

$$\Psi_N(s) := \sum_{\underline{a} \in C'_N} e^{-sn_{\underline{a}}c/3}.$$

For $t \in T \sqcup T^\square$, let $\mathfrak{M}(t)$ be the set of words $b \in \mathfrak{M}$ which are suitable from S^t . The word b can be of the form $\square_\pm(c_i - c_{i+1})$ or $\underline{a}' \cdot \square_\pm(c_i - c_{i+1})$ with $n_{c_i} \geq \Xi n_{\underline{a}'}$. In both cases the order is at least $M + 1$. In the first case or in the second case with \underline{a}' fixed, there are only two possible parabolic pieces for each order. Consequently:

$$\sum_{b \in \mathfrak{M}(t)} e^{-sn_b \frac{c}{3}} \leq \sum_{n \geq M+1} 2e^{-sn \frac{c}{3}} + \sum_{n \geq 1} 2\text{Card}\{\underline{a}' \in \mathfrak{A}^{(\mathbb{N})} : n_{\underline{a}'} = n, t \cdot \underline{a}' \in T \sqcup T^\square\} \sum_{m \geq \Xi n} e^{-sm \frac{c}{3}}.$$

By Lemma 5.2, it comes:

$$\sum_{b \in \mathfrak{M}(t)} e^{-sn_b \frac{c}{3}} \leq \frac{2e^{-sM \frac{c}{3}}}{1 - e^{-s \frac{c}{3}}} + \sum_{n \geq 1} 2^{n+1} \frac{e^{-s\Xi n \frac{c}{3}}}{1 - e^{-s \frac{c}{3}}}.$$

For $s = M^{-1/2}$, since M is large and $\Xi = e^{\sqrt{M}}$, we get:

$$\sum_{b \in \mathfrak{M}(t)} e^{-sn_b \frac{c}{3}} \leq M e^{-\sqrt{M} \frac{c}{3}}.$$

At the power N , this gives:

$$\Psi_N(s) \leq \text{Card}\{\underline{a}' \in \mathfrak{R} : n_{\underline{a}'} \leq N\} \cdot M^N e^{-N\sqrt{M} \frac{c}{3}}.$$

By Lemma 5.2, it comes:

$$\Psi_N(s) \leq 2^N M^N e^{-N\sqrt{M} \frac{c}{3}}.$$

And so:

$$\sum_{\underline{a} \in C_N} e^{-sn_{\underline{a}} \frac{c}{3}} \leq \sum_{n \geq N} 2^n M^n e^{-n\sqrt{M} \frac{c}{3}} \leq e^{-M^{1/4} N}.$$

□

We remark that $(\text{int } Y_{\underline{a}})_{\underline{a} \in C_N}$ is a covering of $\mathcal{E}'_{0,N}$. As we want a covering of \mathcal{E}'_N , we shall study the geometry of $f^n(Y_{\underline{a}})$ for $n \geq 0$.

Lemma 5.11. *For every $\underline{a} \in \mathfrak{R}$, the diameter of $f^n(Y_{\underline{a}})$ is smaller than $\theta^n + |Y_e|e^{-\frac{c}{3}(n_{\underline{a}}-n)}$, for every $n \leq n_{\underline{a}}$.*

Proof. By the first statement of Lemma 3.9, for every $\underline{a} \in \mathfrak{R}$, for every $x \in Y_{\underline{a}}$, the length of the segment S equal to the intersection of $Y_{\underline{a}}$ with the horizontal line passing through x is less than $|Y_e|e^{-\frac{c}{3}n_{\underline{a}}}$. Moreover, the image of this segment by f^n has length less than $|Y_e|e^{-\frac{c}{3}(n_{\underline{a}}-n)}$.

By the second statement of Lemma 3.9, every $y \in Y_{\underline{a}}$ belongs to a curve \mathcal{C} intersecting S , the image of \mathcal{C} by f^n being of length less than θ^n for every $n \leq n_{\underline{a}}$. Thus, the distance between $f^n(x)$ and $f^n(y)$ is smaller than $|Y_e|e^{-\frac{c}{3}(n_{\underline{a}}-n)} + \theta^n$. \square

An immediate consequence is:

Corollary 5.12. *For every $\underline{a} \in \mathfrak{R}$, and $n \in \left[\frac{cn_{\underline{a}}}{6|\log\theta|}, \frac{n_{\underline{a}}}{2}\right]$, the diameter of $f^n(Y_{\underline{a}})$ is less than $(1 + |Y_e|)e^{-n_{\underline{a}}\frac{c}{6}}$.*

We are now ready to compute the Hausdorff dimension of \mathcal{E}'_N .

Proposition 5.13. *The Hausdorff dimension of \mathcal{E}'_N is smaller than $3M^{-1/2}$.*

Proof. Let $\eta > 0$ and let N be sufficiently large so that $(1 + |Y_e|)e^{-N\frac{c}{6}} < \eta$.

Let $x \in \mathcal{E}'_N$, there exists a sequence $(e_i, n_i)_i \in (E'_{0,N} \times \mathbb{N})^{\mathbb{N}}$ such that x belongs to $f^{n_i}(W_{e_i}^s)$ and $n_i \rightarrow \infty$ when i approaches infinity.

Let $e_i =: \underline{a}' \cdot b_1 \cdots b_j \cdots$, with $b_l \in \mathfrak{M}$ for every l and $\underline{a}' \in \mathfrak{R}$ such that $0 < n_{\underline{a}'} \leq N$.

We will show below the following:

Lemma 5.14. *For every $j \geq 1$, $n_{b_j} \leq \Xi n_{\underline{a}' \cdot b_1 \cdots b_{j-1}}$ and so $n_{\underline{a}' \cdot b_1 \cdots b_N} \leq \sum_{i=0}^N \Xi^i n_{\underline{a}'}$.*

Let i be such that $n_i > \sum_{j=0}^N \Xi^j N$ and so $n_i > n_{\underline{a}' \cdot b_1 \cdots b_N}$. Let $j \geq N$ be minimal such that:

$$2n_i \leq n_{\underline{a}' \cdot b_1 \cdots b_j}.$$

By minimality $n_{\underline{a}' \cdot b_1 \cdots b_{j-1}} < 2n_i$. By Lemma 5.14:

$$n_{\underline{a}' \cdot b_1 \cdots b_j} \leq (1 + \Xi)n_{\underline{a}' \cdot b_1 \cdots b_{j-1}} \leq 2(1 + \Xi)n_i.$$

Therefore, the integer $n_{\underline{a}' \cdot b_1 \cdots b_j}$ belongs to $[2n_i, 2(1 + \Xi)n_i]$. That is to say, the point x belongs to $f^{n_i}(Y_{\underline{a}})$ with $\underline{a} := \underline{a}' \cdot b_1 \cdots b_j$ and $n_i \in \left[\frac{n_{\underline{a}}}{2(\Xi+1)}, \frac{n_{\underline{a}}}{2}\right]$. As \underline{a} belongs to C_N , the following family is a covering of \mathcal{E}'_N :

$$\left\{ f^n(Y_{\underline{a}}) : n \in \left[\frac{cn_{\underline{a}}}{6|\log\theta|}, \frac{n_{\underline{a}}}{2}\right], \underline{a} \in C_N \right\}.$$

By Corollary 5.12, the diameter of the boxes of the covering is smaller than η . Let us compute:

$$\Psi_N(s) := \sum_{n \in \left[\frac{cn_{\underline{a}}}{6|\log\theta|}, \frac{n_{\underline{a}}}{2}\right], \underline{a} \in C_N} \text{diam}(f^n(Y_{\underline{a}}))^{3s}.$$

By Corollary 5.12, $\text{diam}(f^n(Y_{\underline{a}})) \leq (1 + |Y_e|)e^{-\frac{\varepsilon}{6}n_{\underline{a}}}$. Thus:

$$\Psi_N(s) \leq \sum_{\underline{a} \in C_N} \frac{n_{\underline{a}}}{2} (1 + |Y_e|)^{3s} e^{-\frac{\varepsilon}{2}n_{\underline{a}}s}.$$

As $\frac{n_{\underline{a}}}{2}(1 + |Y_e|)^{3s}$ is smaller than $e^{\frac{\varepsilon}{6}n_{\underline{a}}s}$ since $s = 1/\sqrt{M}$ and $n_{\underline{a}} \geq M$, it comes that:

$$\Psi_N(s) \leq \sum_{\underline{a} \in C_N} e^{-\frac{\varepsilon}{3}n_{\underline{a}}s}.$$

By Lemma 5.9, we have $\Psi_N(s) \leq \lambda^N$, for $s = M^{-1/2}$ and $\lambda = e^{-M^{1/4}}$. Thus the Hausdorff dimension of \mathcal{E}'_{N_0} is less than $3s = 3M^{-1/2}$. \square

Proof of Lemma 5.14. We show this by induction on N . Let $a_1 \cdots a_l$ be the \mathfrak{A} -spelling of $\underline{a}' \cdot b_1 \cdots b_N$. Let $k < l$ be such that $a_k \cdots a_l$ is the \mathfrak{A} -spelling of b_N . As b_N belongs to \mathfrak{M} , $M + \Xi n_{a_k \cdots a_{l-1}} < n_{a_l}$. Thus $n_{b_N} < (1 + \Xi^{-1})n_{a_l}$.

As e_i is $\sqrt{\Xi}$ -regular, it holds $n_{a_l} \leq M + \sqrt{\Xi}n_{a_1 \cdots a_{l-1}}$.

Thus $n_{a_l} \leq M + \sqrt{\Xi}n_{\underline{a}' \cdot b_1 \cdots b_{N-1}} + \Xi^{-1/2}n_{a_l}$ and so:

$$n_{b_N} \leq \frac{1 + \Xi^{-1}}{1 - \Xi^{-1/2}}(M + \sqrt{\Xi}n_{\underline{a}' \cdot b_1 \cdots b_{N-1}}) < \Xi n_{\underline{a}' \cdot b_1 \cdots b_{N-1}}.$$

\square

6 Properties of the semi-conjugacy

In this section we show that the map i defined in Proposition 4.12 is a bijection, and so that its inverse h is well defined. Then we show that i is Hölder continuous.

We recall that $\check{\mathcal{R}} := \cap_{n \geq 0} F^n(\check{\mathcal{R}})$ and that $F : z \in \mathcal{R} \mapsto f^{N_{\underline{a}(z)}}(z) \in \mathcal{R}$. The following is not obvious and fundamental to prove the injectivity of i :

Proposition 6.1. *The map F is the first return map of $\check{\mathcal{R}}$ in $\check{\mathcal{R}}$ induced by f :*

$$\forall z \in \check{\mathcal{R}}, \forall i \in (0, N_{\underline{a}(z)}), \quad f^i(z) \notin \check{\mathcal{R}}$$

Remark 6.2. The first return of \mathcal{R} in \mathcal{R} is in general not F . Suppose there exists $x \in \mathcal{R}$ such that $\underline{a}(x)$ is of the form $s \square(c_1 - c_2) \cdot s \square(c_1 - c_2) \cdots s \square(c_1 - c_2) \cdots$ with $s \in \mathfrak{Y}_0$ and c_1, c_2 of depth 1 and 2. Then the first return time of x in \mathfrak{R} is $n_s + M + 1$ and not $n_s + M + 1 + n_{c_1}$ as given by F .

6.1 Bijectivity of i

Proof of injectivity of i Let $\underline{w}, \underline{w}' \in \Omega'_G$ be such that $i(\underline{w}) = x = i(\underline{w}')$. By commutativity of the diagram of Proposition 4.8, we can suppose that x belongs to $\check{\mathcal{R}}$, even if it means looking at an iterate. Proposition 6.1 implies that the zero-coordinates of $I^{-1}(\underline{w})$ and $I^{-1}(\underline{w}')$ are of the forms $(g_0, 0)$ and $(g'_0, 0)$. As $(\mathcal{R}_g)_g$ is a partition of \mathcal{R} , it comes that $g_0 = g'_0$. Thus the i -coordinates of $I^{-1}(\underline{w})$ and $I^{-1}(\underline{w}')$ are both equal to (g_0, i) for every $i < n_{g_0}$. Then the n_{g_0} -coordinates of $I^{-1}(\underline{w})$

and $I^{-1}(\underline{w}')$ are both equal to a certain $(g_1, 0)$, with $g_1, g'_1 \in \mathfrak{B}$. Using again this partition property for $f^{n_{g_0}}(x)$, it comes that $g_1 = g'_1$. And so on, we show that the i -coordinates of $\underline{w}, \underline{w}'$ are equal for every $i \geq 0$. By commutativity of the diagram, for every $n \geq 0$, $i(\sigma^{-n}(\underline{w}))$ and $i(\sigma^{-n}(\underline{w}'))$ are also equal. Applying the above argument, it comes that $I^{-1}(\underline{w})$ is equal to $I^{-1}(\underline{w}')$ and so that i is injective since I is bijective.

Surjectivity of i By commutativity of the diagram of Proposition 4.8, it is sufficient to prove that the image of i contains every point $z \in \check{\mathcal{R}}$. Let us construct $\underline{w} \in \Omega'_G$ such that $i(\underline{w}) = z$. By Proposition 6.1, the map $F: \check{\mathcal{R}} \rightarrow \check{\mathcal{R}}$ is bijective. Let $(z_i)_i$ be the preorbit of z by F . We note that there exists $g_i \in \mathfrak{B}$ such that $\underline{a}(z_i) = g_i \cdot \underline{a}(z_{i+1})$ for $i < 0$. Also the sequence $\underline{w} \in \Omega'_G$ associated to $\cdots g_i \cdots \underline{a}(z_0)$ satisfies $i(\underline{w}) = z$.

The proof of Proposition 6.1 is combinatorial and geometric. It needs a few notions.

6.2 Definition and properties of the division

A useful tool introduced in [Ber11] is the Right division on the words \mathfrak{G} .

First let us remark that for a common piece $c_j = \alpha_1 \star \cdots \star \alpha_j$ such that each α_i is in \mathcal{Y}_k , we can associate the concatenation of the \mathfrak{A} -spelling of each $\alpha_i \in \mathcal{Y}$ given by (SR_3) . Thus c_j is canonically associated to a (regular) word \underline{c}_j in the alphabet \mathfrak{A} . As all the common pieces are attached to the same curve $S^\#$, this association is injective.

We say that $\underline{a} \in \mathfrak{G}$ is (*right*) *divisible* by $\underline{a}' \in \mathfrak{G}$ and note $\underline{a}/\underline{a}'$ if one of the following conditions hold:

$$(D_1) \quad \underline{a} = \underline{a}' \text{ or } \underline{a}' = e,$$

$$(D_2) \quad \underline{a} \text{ is of the form } \square_{\pm}(c_l - c_{l+1}) \text{ and satisfies } \underline{c}_l/\underline{a}', \text{ with } \underline{c}_l \text{ the } \mathfrak{A}\text{-spelling of } c_l,$$

$$(D_3) \quad \text{there are splittings } \underline{a} = \underline{a}_3 \cdot \underline{a}_2 \cdot \underline{a}_1 \text{ and } \underline{a}' = \underline{a}'_2 \cdot \underline{a}_1 \text{ into words } \underline{a}_1, \underline{a}_2, \underline{a}'_2, \underline{a}_3 \in \mathfrak{A}^{(\mathbb{N})} \text{ such that } \underline{a}_2/\underline{a}'_2 \text{ and } n_{\underline{a}_3} + n_{\underline{a}_1} \geq 1.$$

The two last conditions are recursive but the recursion decreases the order $n_{\underline{a}}$. Thus the right divisibility is well defined by induction on $n_{\underline{a}}$. In Proposition 5.14 of [Ber11], we showed:

Proposition 6.3. *The right divisibility $/$ is an order relation on \mathfrak{G} . Moreover for all $\underline{a}, \underline{a}', \underline{a}'' \in \mathfrak{G}$:*

1. *If $\underline{a}/\underline{a}'$ then $n_{\underline{a}} \geq n_{\underline{a}'}$, with equality iff $\underline{a} = \underline{a}'$.*
2. *If $\underline{a}/\underline{a}'$, $\underline{a}/\underline{a}''$ and $n_{\underline{a}'} \geq n_{\underline{a}''}$ then $\underline{a}'/\underline{a}''$.*

Remark 6.4. If $\underline{a} \in \mathfrak{G}$ is divided by $b \in \mathfrak{A} \setminus \mathfrak{Y}_0$ such that $\underline{a} = \underline{a}' \cdot b$.

A first application of this division is related to the C^1 distance between flat stretched curves. Let us define the C^1 -distance between two flat stretched curves S and S' . Let $\rho, \rho' \in C^{1+Lip}([0, 1], \mathbb{R})$ be the functions with respective graph sent by y_e^0 to S and S' respectively. The distance between S and S' is then:

$$d(S, S') := \sup_x |\rho(x) - \rho'(x)| + \sup_x |d\rho(x) - d\rho'(x)|$$

The first application is the following:

Proposition 6.5 (Prop. 5.17, Prop. 6.1 of [Ber11]). *For every $t \cdot \underline{a}, t' \cdot \underline{b} \in T \sqcup T^\square$, if there exists $d \in \mathfrak{G}$ such that:*

$$\underline{a}/\underline{d} \quad \text{and} \quad \underline{b}/\underline{d}$$

then the curves $S^{t \cdot \underline{a}}$ and $S^{t' \cdot \underline{b}}$ are $b^{n_d/4}$ -close in the C^1 -topology.

Furthermore the same parabolic symbols of depth $j \leq \Xi(n_d + M + 1)$ are suitable from $S^{t \cdot \underline{a}}$ and $S^{t' \cdot \underline{b}}$.

An application of this proposition is the following:

Corollary 6.6. *If $s \cdot g$ belongs to \mathfrak{B} , then $s' \cdot g'$ belongs to \mathfrak{B} for every $s, s' \in \mathfrak{Y}_0$.*

Proof. Let $g_1 \cdots g_n$ be the \mathfrak{A} -spelling of g' . By Remark 3.10, the interior of the box $Y_{g'}$ intersects both $S^{t \cdot s}$ and $S^{t \cdot s'}$. As by Proposition 6.5, $S^{t \cdot s'}$ belongs to $D(g_1)$, the word g_1 is suitable from $S^{t \cdot s'}$. Combining inductively these two arguments, it comes that $g' = g_1 \cdots g_n$ is suitable from $S^{t \cdot s'}$. By Proposition 4.2, this implies that $s' \cdot g'$ belongs to \mathfrak{B} . \square

A geometric consequence of the division is the following Lemma:

Lemma 6.7. *If $\underline{a}/\underline{b}$ with $\underline{a}, \underline{b} \in \mathfrak{R}$, then $f^{n_{\underline{a}}}(Y_{\underline{a}}) \subset f^{n_{\underline{b}}}(Y_{\underline{b}})$.*

Proof. We proceed by induction on $n_{\underline{a}}$. If $\underline{a} \in \mathfrak{Y}_0$ then $\underline{b} \in \{\underline{a}, e\}$. The inclusion is clear. Otherwise $\underline{a} = \underline{a}_1 \cdot \underline{a}_2$, with $\underline{a}_1 \in \mathfrak{R}$ and $\underline{a}_2 \in \mathfrak{A}$. There are two possibilities:

If $n_{\underline{a}_2} \geq n_{\underline{b}}$, then $\underline{a}_2/\underline{b}$ by Proposition 6.3.2. Either $\underline{a}_2 \in \mathfrak{Y}_0$ and $\underline{b} \in \{\underline{a}, e\}$ from which the inclusion is clear; either \underline{a}_2 is of the form $\square_{\pm}(c_i - c_{i+1})$ and, since $\underline{b} \in \mathfrak{R}$, it holds $\underline{c}_i/\underline{b}$. As $\underline{c}_i \in \mathfrak{R}$, by induction $f^{n_{\underline{c}_i}}(Y_{\underline{c}_i}) \subset f^{n_{\underline{b}}}(Y_{\underline{b}})$. As $f^{M+1+n_{\underline{a}_1}}(Y_{\underline{a}_1}) \subset Y_{\underline{c}_i}$, it holds $f^{n_{\underline{a}}}(Y_{\underline{a}}) \subset f^{n_{\underline{b}}}(Y_{\underline{b}})$.

If $n_{\underline{a}_2} < n_{\underline{b}}$, then $\underline{b} = \underline{b}_1 \cdot \underline{a}_2$ with $\underline{b}_1 \in \mathfrak{R}$ and $\underline{a}_1/\underline{b}_1$ by Proposition 6.3.2 and by definition of the division. By induction, $f^{n_{\underline{a}_1}}(Y_{\underline{a}_1}) \subset f^{n_{\underline{b}_1}}(Y_{\underline{b}_1})$. From the geometries of $f^{n_{\underline{b}_1}}(Y_{\underline{b}_1})$ and $Y_{\underline{a}_2}$, a same argument as for the proof of Proposition 3.7 shows that the subset $Y_{\underline{b}_1} \cap f^{-n_{\underline{b}_1}}(Y_{\underline{a}_2})$ is connected and equal to $Y_{\underline{b}}$. Thus $f^{n_{\underline{a}_1}}(Y_{\underline{a}_1}) \cap Y_{\underline{a}_2} \subset f^{n_{\underline{b}_1}}(Y_{\underline{b}})$. This implies that $f^{n_{\underline{a}}}(Y_{\underline{a}}) \subset f^{n_{\underline{b}}}(Y_{\underline{b}})$. Looking at the image by $f^{n_{\underline{a}_2}}$ of this inclusion, we finish the proof. \square

6.3 Proof of the first return time property (Proposition 6.1)

Let $x \in \tilde{\mathcal{R}} \cap_{n \geq 0} F^n(\mathcal{R})$ and let $\underline{a}(x) = a_1 \cdots a_i \cdots$ be its \mathfrak{A} -spelling. We recall that $N_{\underline{a}(x)}$ is the first return time of $\underline{a}(x)$ into \tilde{R} by the shift $\tilde{\sigma}$ of $\mathfrak{A}^{\mathbb{N}}$. Put $g' := a_1 \cdots a_{N_{\underline{a}(x)}-1}$ and $n' := n_{g'}$. Let n be the first return time of x in $\tilde{\mathcal{R}}$. We want to show that n is equal to n' . We notice that $n \leq n'$. Put $x' := f^n(x)$. These notations with respect to the time scale are depicted figure 5.

1) We suppose that there exists i such that $n = n_{a_1 \cdots a_i}$. Put $\underline{a}' = a'_1 \cdots a'_i \cdots := \underline{a}(x')$. By Proposition 4.1, $a_1 \cdots a_i \cdot a'_1 \cdots a'_j$ belongs to \mathfrak{R} for every $j \geq 0$ since $f^n(Y_{a_1 \cdots a_i}) \cap Y_{a'_1 \cdots a'_j}$ contains $x' \notin \partial^u Y_e \cup W^s(A)$ and the latter set contains the boundary of $Y_{a'_1 \cdots a'_j}$. Thus $a_1 \cdots a_i \cdot \underline{a}'$ belongs to \tilde{R} and x to $W_{a_1 \cdots a_i \underline{a}'}^s$. By uniqueness of the \mathfrak{A} -spelling, it comes that $a_{i+k} = a'_k$ for every k , and so $n = N_{\underline{a}} = n'$.

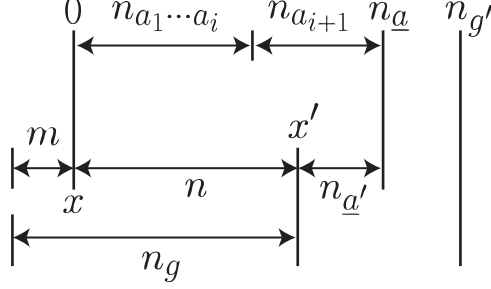


Figure 5: Notations for the proof of Proposition 6.1.

2) Suppose for the sake of contradiction that $n \in (n_{a_1 \dots a_i}, n_{a_1 \dots a_{i+1}})$, for a certain i . Put $\underline{a} = a_1 \dots a_{i+1}$. Note that the symbol a_{i+1} is parabolic since it cannot be simple by \hat{G}_1 . Let $g \in \mathfrak{B}$ be such that $x' := f^n(x)$ belongs to $f^{n_g}(\mathcal{R}_g)$. The minimality of n implies $n_g \geq n$. If $n = n_g$ then $x \in \mathcal{R}_g$ and $n' = n_g = n$. This contradicts $n \in (n_{a_1 \dots a_i}, n_{a_1 \dots a_{i+1}})$. Thus $n_g > n$.

We are going to show that $n = n_{a_1 \dots a_i}$ which is a contradiction. For this end, we apply the following Lemma 6.8 (shown below) with $\underline{a} := a_1 \dots a_{i+1}$ and \underline{b} some first letters of $\underline{a}(f^n(x))$:

Lemma 6.8. *Let $\underline{a}, \underline{b} \in \mathfrak{R}$ and let $n \in [0, n_{\underline{a}}]$ be such that $f^n(Y_{\underline{a}}) \cap \text{int } Y_{\underline{b}} \neq \emptyset$ and $n_{\underline{b}} + n \geq n_{\underline{a}}$. Then there exists $\underline{a}' \in \mathfrak{R} \cup \{e\}$ such that:*

$$\underline{a}/\underline{a}' \quad \text{and} \quad n + n_{\underline{a}'} = n_{\underline{a}},$$

and \underline{b} starts by \underline{a}' (i.e. $\underline{b} = \underline{a}' \cdot \underline{b}'$, with $\underline{b}' \in \mathfrak{A}^{(\mathbb{N})} \cup \{e\}$).

We remark that a_{i+1}/\underline{a}' by Proposition 6.3.2. We are going to show that \underline{a}'/a_{i+1} , which implies $a_{i+1} = \underline{a}'$ and so $n = n_{\underline{a}} - n_{a_{i+1}} = n_{a_1 \dots a_i}$, the contradiction.

By the above Lemma, \underline{a}' belongs to \mathfrak{R} . By Lemma 6.7, $f^n(Y_{\underline{a}}) \subset Y_{\underline{a}'}$, thus $f^{n_g}(Y_g) \cap Y_{\underline{a}'}$ contains $x' = f^n(x)$. By Proposition 4.1, $g \cdot \underline{a}'$ belongs to \mathfrak{R} .

To prove that \underline{a}'/a_{i+1} , we are going to apply the same Lemma 6.8, with $g \cdot \underline{a}'$ instead of \underline{a} , with $m := n_g - n$ instead of n , and \underline{a} instead of \underline{b} . We can apply this Lemma since $g \cdot \underline{a}'$ and \underline{a} belong to \mathfrak{R} .

It remains to observe that $m = n_{g \cdot \underline{a}'} - n_{\underline{a}}$ (see fig. 5) and that the intersection of $f^m(Y_{g \cdot \underline{a}'})$ with the interior $Y_{\underline{a}}$ contains the point x .

Lemma 6.8 gives $g \cdot \underline{a}'/\underline{a}$. As a_{i+1} is a parabolic symbol and $\underline{a}' \neq e$, it comes \underline{a}'/a_{i+1} from Remark 6.4. The expected contradiction.

Proof of Lemma 6.8. We proceed by induction on $n_{\underline{a}}$ to show the existence of such an $\underline{a}' \in \mathfrak{R}$.

If $\underline{a} \in \mathfrak{Y}_0$ then either $n = 0$ or $n = n_{\underline{a}}$; take $\underline{a}' = \underline{a}$ or e respectively.

The case $\underline{a} = \square_{\pm}(c_i - c_{i+1})$ does not occur since $\underline{a} \in \mathfrak{R}$.

Let $\underline{a} = \underline{a}_1 \cdot a_2$ be with $a_2 \in \mathfrak{A}$. If $n > n_{\underline{a}_1}$, then either $a_2 \in \mathfrak{Y}_0$ and $n = n_{\underline{a}}$; either $a_2 = \square_{\pm}(c_i - c_{i+1})$ and we can use the induction hypothesis on $\underline{c}_i \in \mathfrak{R}$.

If $n \leq n_{\underline{a}_1}$, then by induction there exists $\underline{a}'_1 \in \mathfrak{R}$ such that $n + n_{\underline{a}'_1} = n_{\underline{a}_1}$ and a_1/\underline{a}'_1 . Furthermore, $\underline{b} \in \mathfrak{R}$ starts by \underline{a}'_1 : there exists $\underline{b}' \in \mathfrak{A}^{(\mathbb{N})}$ such that $\underline{b} = \underline{a}'_1 \cdot \underline{b}'$.

Let $b_2 \in \mathfrak{A}$ be the first letter of \underline{b}' . We want to show that $b_2 = a_2$ since $\underline{a}' = \underline{a}'_1 \cdot a_2$ divides \underline{a} and satisfies $n + n_{\underline{a}'} = n_{\underline{a}}$.

As \underline{b} belongs to \mathfrak{R} , the curve $S^{\# \cdot \underline{a}'}$ belongs to the domain $D(b_2)$ and $n_{b_2} \leq M + \Xi n_{\underline{a}'}$.

As $\underline{a}_1 / \underline{a}'_1$, by Proposition 6.5, the parabolic pieces of $S^{\# \cdot \underline{a}_1}$ and $S^{\# \cdot \underline{a}'_1}$ with order less than $M + \Xi n_{\underline{a}'_1}$ are pairwise the same. In particular, $S^{\# \cdot \underline{a}_1}$ belongs to the domain $D(b_2)$ and $b_2 \in \mathfrak{P}(\# \cdot \underline{a}_1)$.

Also $f^n(Y_{\underline{a}})$ intersects the interior of $Y_{\underline{b}} \subset Y_{\underline{a}'_1 \cdot b_2}$ and

$$f^n(Y_{\underline{a}}) \cap Y_{\underline{a}'_1 \cdot b_2} = f^n(Y_{\underline{a}_1} \cap f^{-n_{\underline{a}_1}}(Y_{a_2})) \cap Y_{\underline{a}'_1} \cap f^{-n_{\underline{a}'_1}}(Y_{b_2}) \subset f^{-n_{\underline{a}'_1}}(Y_{a_2} \cap Y_{b_2}).$$

Thus the interior of $Y_{a_2} \cap Y_{b_2}$ is not empty and so the partition property of $\mathcal{P}(\# \cdot \underline{a}'_1)$ implies $a_2 = b_2$. \square

6.4 Hölder continuity of i

Let us show that the map i is $2e^{-c/3}$ -Hölder. Let $\underline{w} = (w_i)_i$, $\underline{w}' = (w'_i)_i \in \Omega'_G$ be such that $d(\underline{w}, \underline{w}') \leq 2^{-n}$. This means that $w_i = w'_i$ for every $|i| \leq n$. We can suppose n maximal with this property. There exists $m \geq n$ such that $w_i = w'_i$ for every $-m \leq i \leq 0$ and such that $w_{-m} = w'_{-m}$ are of the form $(g', 0)$ with $s \cdot g' \in \mathfrak{B}$ for $s \in \mathfrak{Y}_0$ or $g = g' \in \mathfrak{Y}_0$.

We remark that there exists $\underline{a} \in \mathfrak{G}$ with $n_{\underline{a}} \geq m + n$ such that $x := i(\underline{w})$ and $x' := i(\underline{w}')$ satisfy:

$$f^{-m}(x) \in Y_{\underline{a}} \ni f^{-m}(x')$$

It is easy to see that $f^{-m}(x)$ and $f^{-m}(x')$ belong to flat stretched curves S^t and $S^{t'}$ respectively, with $t, t' \in T \sqcup T^\square$ such that $t \cdot \underline{a}$ and $t' \cdot \underline{a}$ belong to $T \sqcup T^\square$.

Unfortunately the words $\underline{a} =: (a_i)_i$ is not Ξ -regular since it does not satisfy (R^ξ) in general, but:

$$(6.1) \quad n_{a_i} \leq \Xi(M + \sum_{j < i} n_{a_j}), \quad \forall i.$$

By Remark 3.10, $f^{-m}(x')$ belongs to a curve $\mathcal{C} \subset Y_{\underline{a}}$ such that:

- the length of $f^k(\mathcal{C})$ is smaller than θ^k for every $k \leq m + n$,
- the curve \mathcal{C} intersects S^t at a point $z \in Y_{\underline{a}}$.

From this it follows that the distance between $f^m(z)$ and x' is smaller than θ^m . Using moreover h -times property of $\underline{a}(S^t)$, the distance between $f^m(z)$ and x is smaller than $|Y_e| \cdot e^{-(n_{\underline{a}} - m)c/3}$. Thus:

$$d(i(\underline{w}), i(\underline{w}')) \leq d(x, x') \leq \theta^n + e^{-nc/3} \leq 2d(\underline{w}, \underline{w}')^{\frac{c}{3 \log 2}}.$$

7 Periodic points with orbit which does not intersect \mathcal{R}

In this section we show the following:

Proposition 7.1. *For every p , the number of fixed points x of f^p with orbit which does not intersect \mathcal{R} is less than $pe^{p/\sqrt{M}} + (M + 1)2^{p/(M+1)}$.*

Proof. Let us first study the periodic points in K_\square . In §5.3, we showed that K_\square is either empty or $f^{M+1}|K_\square$ is conjugated to the shift on $\{0,1\}^\mathbb{Z}$. The n^{th} -iterate of the latter shift has 2^n fixed points. Therefore:

$$\text{Card Fix}(f^n|K_\square) \leq (M+1)2^{n/(M+1)}$$

Let us now study the periodic points x in $\tilde{\mathcal{R}} \setminus \mathcal{R}$. Let $(a_i)_i := \underline{a}(x)$. Let $p \geq 1$ be such that $f^p(x) = x$. We are going to show that $\underline{a}(x)$ is preperiodic.

If there exists j such that $p = n_{a_1 \dots a_j}$ then by Proposition 4.1, $a_{n+j} = a_n$ for every n . This contradicts that $x \notin \mathcal{R}$.

Consequently there exists $j \geq 0$ such that $p \in (n_{a_1 \dots a_{j-1}}, n_{a_1 \dots a_j})$. By Lemma 6.8, there exists $i < j$ such that:

$$a_1 \dots a_j / a_1 \dots a_i \quad \text{and} \quad p + n_{a_1 \dots a_i} = n_{a_1 \dots a_j}$$

As $n_{a_1 \dots a_{j-1}} < p < n_{a_1 \dots a_j}$, this implies that $a_j / a_1 \dots a_i$.

Using the same Lemma, it holds that there exists $i' > i$ such that

$$a_1 \dots a_{j+1} / a_1 \dots a_{i'} \quad \text{and} \quad p + n_{a_1 \dots a_{i'}} = n_{a_1 \dots a_{j+1}}$$

$$\Rightarrow a_{j+1} / a_{i+1} \dots a_{i'} \quad \text{and} \quad a_1 \dots a_{j+1} / a_1 \dots a_{i'} \quad \text{and} \quad n_{a_{i+1} \dots a_{i'}} = n_{a_{j+1}}$$

By Proposition 6.3, it comes that $a_{i+1} \dots a_{i'} = a_{j+1}$. By uniqueness of the \mathfrak{A} -spelling, $a_{j+1} = a_{i+1}$.

And so on, $a_{i+k} = a_{j+k}$, for every $k \geq 1$. From this it comes that $(a_i)_{i \geq 1}$ is equal to the preperiodic sequence:

$$a_1 \cdot a_2 \dots a_i \cdot a_{i+1} \cdot a_{i+2} \dots a_{j-1} \cdot a_j \cdot a_{i+1} \cdot a_{i+2} \dots a_{j-1} \cdot a_j \cdot a_{i+1} \cdot a_{i+2} \dots a_{j-1} \cdot a_j \dots$$

The periodic chain is $\underline{b} := a_{i+1} \cdot a_{i+2} \dots a_{j-1} \cdot a_j$. It must be very irregular:

Lemma 7.2. *There exists $b_1 \dots b_k \in \mathfrak{M}^{(\mathbb{N})}$ equal to \underline{b} .*

Proof. The word \underline{b} cannot be in \mathfrak{R} since otherwise x would be in \mathcal{R} . Let $b_1, \underline{b}' \in \mathfrak{G}$ be such that $\underline{b} = b_1 \cdot \underline{b}'$, $b_1 \notin \mathfrak{R}$ and suppose b_1 with an order minimal with this property. Note that \underline{b}' can be empty but b_1 not. If \underline{b}' is empty then we are done, since by minimality b_1 belongs to \mathfrak{M} .

If \underline{b}' is not empty, assume for the sake of contradiction that it belongs to \mathfrak{R} . We recall that $a_j / a_1 \cdot a_2 \dots a_i$ and so $n_{\underline{b}'} \geq n_{a_j} \geq n_{a_1 \cdot a_2 \dots a_i}$. As, by Ξ -regularity, $n_{a_{i+1}}$ is at most $M + \Xi n_{a_1 \cdot a_2 \dots a_i}$, it is also at most $M + \Xi n_{\underline{b}'}$. Furthermore $a_{j+1} = a_{i+1}$, and so $\underline{b}' \cdot a_{i+1}$ is Ξ -regular. Doing the same argument we show that $\underline{b}' \cdot a_{i+1} \dots a_m$ is Ξ -regular. Thus $\underline{b}' \cdot a_{i+1} \dots a_m \dots$ belongs to \tilde{R} and \underline{a} belongs to R . A contradiction.

Thus we can write \underline{b} in the form $b_1 \cdot b_2 \cdot \underline{b}''$ with $b_1, b_2 \in \mathfrak{M}$. Again by the full argument, we show that $\underline{b}'' \notin \mathfrak{R}$, and so on, it follows that \underline{b} belongs to $\mathfrak{M}^{(\mathbb{N})}$. \square

The above Lemma defines the following canonical map:

$$\underline{b}: x \in \text{Per}_f \cap \tilde{\mathcal{R}} \setminus \mathcal{R} \mapsto b_1 \dots b_k \in \mathfrak{M}^{(\mathbb{N})}.$$

Actually the image of \underline{b} is included in $\mathfrak{M}^{(\mathbb{N})} \cap \mathfrak{G}$. Let us prove the following:

Lemma 7.3. *If $\underline{b}(x) = \underline{b}(x')$ for $x, x' \in \text{Per}_f \cap \tilde{\mathcal{R}} \setminus \mathcal{R}$, then the periodic orbits of x and x' are equal.*

Proof. Let $\underline{a}(x) = \underline{a} \cdot \underline{b} \cdots \underline{b} \cdots$ and $\underline{a}(x') = \underline{a}' \cdot \underline{b} \cdots \underline{b} \cdots$. For $l \geq 0$, put:

$$\underline{a}_l := \underline{a} \cdot \underbrace{\underline{b} \cdots \underline{b}}_{l \text{ times.}} \quad \text{and} \quad \underline{a}'_l := \underline{a}' \cdot \underbrace{\underline{b} \cdots \underline{b}}_{l \text{ times.}}$$

By Lemma 3.9, the points $f^{n_{\underline{a}_l}}(x)$ and $f^{n_{\underline{a}'_l}}(x')$ are $\theta^{ln_{\underline{b}}}$ -close to respectively the curves $S^{\# \cdot \underline{a}_l}$ and $S^{\# \cdot \underline{a}'_l}$. Moreover, as $\underline{a}_l/\underline{b} \cdots \underline{b}$ and $\underline{a}'_l/\underline{b} \cdots \underline{b}$ with $n_{\underline{b} \cdots \underline{b}} = l \cdot n_{\underline{b}}$, by Proposition 6.5, the flat stretched curves $S^{\# \cdot \underline{a}_l}$ and $S^{\# \cdot \underline{a}'_l}$ are $C^1\text{-}b^{l \cdot n_{\underline{b}}/4}$ close. For the same reason, the sequences $(S^{\# \cdot \underline{a}_l})_l$ and $(S^{\# \cdot \underline{a}'_l})_l$ are Cauchy and so both converge to a same flat stretched curve S .

As any domain of the form $D(\underline{b} \cdots \underline{b})$ is closed and contains the sequence $(S^{\# \cdot \underline{a}_l})_l$, it contains its limit S . By Lemma 3.9, the distance between $f^{n_{\underline{a}_l}}(x)$ and $S^{\# \cdot \underline{a}_l}$ approaches 0 as l approaches infinity. Thus $f^{n_{\underline{a}_l}}(x)$ approaches $S_{\underline{b} \cdots \underline{b}}$ as l approaches infinity, and likewise for $f^{n_{\underline{a}'_l}}(x')$.

As x, x' are periodic points, for l sufficiently large, $f^{n_{\underline{a}_l}}(x)$ and $f^{n_{\underline{a}'_l}}(x')$ belong to:

$$\bigcap_{n \geq 0} S_{\underbrace{\underline{b} \cdots \underline{b}}_{n \text{ times.}}}$$

which is a single point. It comes that $f^{n_{\underline{a}_l}}(x)$ and $f^{n_{\underline{a}'_l}}(x')$ are equal. \square

Thus we shall prove that the cardinality of $\{\underline{b} : \mathfrak{M}^{(\mathbb{N})}, n_{\underline{b}} = p\}$ grows exponentially slowly with p . This begins with:

Lemma 7.4. *The cardinality of $\{\square_{\pm}(c_{l-1}(t) - c_l(t)) : t \in T \sqcup T^{\square}\}$ is at most $2^{2+\Xi^{-2}l}$, for every $l \geq 0$.*

Proof. Let $\tilde{T}_l := \{\# \cdot d \in T \sqcup T^{\square} : d \in \mathfrak{G}, n_d \leq \Xi^{-2}l\}$. By Lemma 5.2, the cardinality of \tilde{T}_l is at most $2^{1+\Xi^{-2}l}$. On the other hand, Proposition 7.5 (ii) of [Ber11] and Proposition 6.5, it holds:

Lemma 7.5. *For every $t \in T \sqcup T^{\square}$, for every $l \geq 2M\Xi^2$, there exists $t' \in \tilde{T}_l$ such that S^t and $S^{t'}$ are $b^{\frac{l}{8M\Xi^2}}\text{-}C^1\text{-distant}$.*

By definition of strong regularity, $S^{t^{\square}}$ and $S^{t'^{\square}}$ must be $\aleph(l)$ -tangent to $Y_{c_l(t)}$ and $Y_{c_l(t')}$ respectively. Either $\underline{c}_l(t)$ and $\underline{c}_l(t')$ are equal or their extensions have disjoint interiors. Also the "fold" of $S^{t^{\square}}$ is at most $e^{-c^+(n_{c_l(t)}+2\aleph(l))} \geq e^{-c^+(Ml+2\aleph(l))}$ close to $\partial^s Y_{c_l(t)}$. By Remark 2.15, for $l \geq 2M\Xi^2$:

$$(7.1) \quad b^{\frac{l}{8M\Xi^2}} e^{c^+(M+1)} < e^{-c^+(Ml+2\aleph(l))}.$$

Thus $c_l(t)$ and $c_l(t')$ are equal. This implies that for $l \geq 2M\Xi^2$:

$$\text{Card} \{\square_{\pm}(c_{l-1}(t) - c_l(t)) : t \in T \sqcup T^{\square}\} = \text{Card} \{\square_{\pm}(c_{l-1}(t) - c_l(t)) : t \in \tilde{T}_l\}$$

Since by Lemma 5.2, the cardinality of \tilde{T}_l is at most $2^{1+l/\Xi^2}$, the above cardinality is at most $2^{2+l/\Xi^2}$.

For $1 \leq l \leq 2M\Xi^2$, as all the flat stretched curves have a Hausdorff distance between each other at most $4\theta = 4/|\log b|$ which is less than $e^{-c^+(M+1)} e^{-c^+(Ml+2\aleph(l))}$ by Remark 2.15, they are all

tangent to the same Y_{c_l} . Thus, for every $l \geq 1$, the cardinality of $\{\square_{\pm}(c_{l-1}(t) - c_l(t)) : t \in T \sqcup T^{\square}\}$ is at most $2^{2+l^2/\Xi}$. \square

This upperbound is useful to show below the following, with $s := 1/\sqrt{M}$:

$$(7.2) \quad \sum_{b \in \mathfrak{M}} e^{-sn_b} \leq \frac{1}{M}$$

This lemma implies that $\sum_{b \in \mathfrak{M}(\mathbb{N})} e^{-sn_b} \leq 1$. Consequently, for every $p \geq 1$, the cardinality of the fixed points of f^p with orbit which does not intersect \mathcal{R} is at most $p \cdot e^{p/\sqrt{M}} + (M+1)2^{p/(M+1)}$. This is the statement of Proposition 7.1.

Let us prove inequality (7.2). We remark that $\sum_{b \in \mathfrak{M}} e^{-sn_b}$ is at most:

$$\sum_{l \geq 1} \text{Card} \{\square_{\pm}(c_{l-1}(t) - c_l(t)) : t \in T \sqcup T^{\square}\} e^{-sn_{\square}(c_{l-1}(t) - c_l(t))} + \sum_{n \geq 1} 2 \text{Card} \{\underline{a} \in \mathfrak{R} : n_{\underline{a}'} = n\} \sum_{m \geq \Xi n} e^{-sm \frac{\epsilon}{3}}$$

By Lemma 7.4 and since $n_{\square_{\pm}(c_{l-1}(t) - c_l(t))} \geq M + l$:

$$\sum_{l \geq 1} \text{Card} \{\square_{\pm}(c_{l-1}(t) - c_l(t)) : t \in T \sqcup T^{\square}\} e^{-sn_{\square}(c_{l-1}(t) - c_l(t))} \leq \sum_{l \geq 1} 2^{2+l\Xi^{-2}} e^{-(M+l)s} \leq \frac{1}{2M}$$

On the other hand, by Lemma 5.2:

$$\sum_{n \geq 1} 2 \text{Card} \{\underline{a} \in \mathfrak{R} : n_{\underline{a}'} = n\} \sum_{m \geq \Xi n} e^{-sm \frac{\epsilon}{3}} \leq \sum_{n \geq 1} 2^{n+1} \frac{e^{-s\Xi n \frac{\epsilon}{3}}}{1 - e^{-\frac{\epsilon}{3}s}} \leq \frac{1}{2M}$$

\square

8 Proof of the analytical ingredients

This section is devoted to show Proposition 3.3, Lemma 3.9, Proposition 4.3 and the postponed argument of Proposition 3.11.

We are going to use concepts of [BC91] and [YW01], that we recall now.

Definition 8.1. Let $D(e_k)$ be the set of points $z \in Y_e$ such that the matrix $T_z f^k$ have a unique direction more contracting, say $e_k(z)$:

$$\frac{\|T_z f^k(e_k(z))\|}{\|e_k\|} < \frac{\|T_z f^k(u)\|}{\|u\|}, \quad \forall u \notin \mathbb{R}e_k(z).$$

The C^1 -vector field e_k is tangent to the leaves of a foliation \mathcal{F}_k .

The set $D(e_k)$ is open and we can assume e_k unitary and continuous. It is then of class C^1 and tangent to the leaves of a foliation \mathcal{F}_k .

Actually the vector field e_k is in general very wild and so \mathcal{F}_k is. To tame this geometry a nice assumption is the following, for $z \in \mathbb{R}^2$:

$$(\mathcal{PCE}^k) \quad \|T_z f^j\| \geq e^{-\Xi c^+(M+1+j)}, \quad j \in [0, k].$$

We notice that if z satisfies (\mathcal{PCE}^k) , then it belongs to $D(e_k)$ by Remark 2.15.

Lemma 8.2 (Lem. 2.1, Cor. 2.1 and 2.2 of [YW01]). *If $z \in \mathbb{R}^2$ satisfies condition (\mathcal{PCE}^k) , then:*

- $\|T_z f^i(e_k)\| \leq b^{i/2}$ for every $0 \leq i \leq k$,
- *There is a neighborhood U of z such that the C^1 -vector field $e_k|_U$ and $e_i|_U$ are $b^{\frac{i}{2}}$ -close, for every $i \leq k$.*

A nice consequence of the above Lemma is the following (Lem. 13.5 [Ber11]):

Lemma 8.3. *Let $z \in Y_e$ be satisfying condition (\mathcal{PCE}^k) . Then the $\mathcal{F}_k|_{Y_e}$ leaf of z is \sqrt{b} - C^1 -close to a component of a curve of the form $\{f_a(x) + y = \text{cst}\} \cap Y_e$. This leaf is $b^{i/2}$ -contracted by f^i for every $i \leq k$.*

Another powerful concept of [BC91] and [YW01] is:

Definition 8.4. A *critical point of order k* of a flat stretched curve S is a point $z \in S \cap D(e_k)$ such that $e_k(z)$ is a tangent to S at z .

A non trivial application of these concepts is the following. Let $\Delta \in \mathfrak{A}$ be a parabolic symbol and let $S \in D(\Delta)$ be a flat stretched curve. Let $\chi_{\Delta(S)}$ be the cone field formed by the vectors $(z, u) \in T\mathbb{R}^2$ such that there exists $z' \in S_\Delta$ which is $\theta^{n_\Delta/3}$ -close to z and there exists $u' \in T_{z'}S_\Delta$ such that $|\angle(u, u')| \leq \theta^{n_\Delta/3}$.

Proposition 8.5 (Prop. 5.9 [Ber11]). *For every $(z, u) \in \chi_{\Delta(S)}$, the vector $T_z f^{n_\Delta}(u)$ belongs to χ and:*

$$\|T_z f^{n_\Delta}(u)\| \geq e^{kc/3} \|T_z f^{n_\Delta-k}(u)\|, \quad \forall k \leq n_\Delta.$$

For $s \in \mathfrak{Y}_0$, let χ_s denote the cone field on Y_s defined by $\chi|_{Y_s}$.

Proof of Proposition 3.3 and Lemma 3.9. During this proof, we are also going to *respectively* show what we need to change to prove Remarks 3.4 and 3.10.

Let $g = a_1 \cdots a_i$ be a sequence of \mathfrak{A} -symbols which is Ξ -regular (resp. satisfies 6.1). By proceeding as for Proposition 3.6 of [Ber11], we can prove that for every $z \in Y_g$, every unit vector $u \in \chi(z)$, $j < i$:

$$(8.1) \quad T_z f^{n_{a_1 \cdots a_j}}(u) \in \chi_{a_{j+1}} \quad \text{and} \quad T_z f^{n_g}(u) \in \chi.$$

By Proposition 8.5 and \hat{G}_4 , it follows that:

$$(8.2) \quad \|T_z f^{n_{a_1 \cdots a_j}}(u)\| \geq e^{n_{a_1 \cdots a_j} c/3}, \quad \forall j \leq i.$$

$$(8.3) \quad \|T_z f^{n_g}(u)\| \geq e^{kc/3} \|T_z f^{n_g-k}(u)\|, \quad \forall k \leq n.$$

This implies obviously the first inequality of Lemma 3.9.1.

Let us show the second inequality of Lemma 3.9.1. Let $k \leq n_g$ and $j < i$ be such that $k \in [n_{a_1 \cdots a_i}, n_{a_1 \cdots a_{i+1}}]$. From (8.2) at $j = i + 1$ and since $\|Tf\| \leq e^{c^+}$, it follows that:

$$\|T_z f^k(u)\| \geq e^{n_{a_1 \cdots a_i} \frac{c}{3}} e^{n_{a_{i+1}} (\frac{c}{3} - c^+)}.$$

As $n_{a_{i+1}} \leq M + \Xi n_{a_1 \dots a_i}$ (resp. $\leq \Xi(M + n_{a_1 \dots a_i})$), it comes:

$$(8.4) \quad \|T_z f^k(u)\| \geq e^{\frac{c}{3}} e^{-(M + \Xi n_{a_1 \dots a_i})c^+} \geq e^{-(M + \Xi k)c^+}, \quad \forall k \leq n_g,$$

which is the second inequality of Lemma 3.9.1 (resp. (\mathcal{PCE}^{n_g}) is satisfied).

As done to prove Proposition 3.6 of [Ber11], one can show that Y_g is a box. This means that Y_g is diffeomorphic to a the filled square $[0, 1]^2$ such that $[0, 1] \times \{0, 1\}$ corresponds to a pair of segments of $\{y = \pm 2\theta\}$, this is the second bullet of Proposition 3.3.

We show below the following lemma:

Lemma 8.6. *Every point in $\partial^s Y_g$ satisfies (\mathcal{PCE}^∞) .*

By Lemma 8.3 with $k = \infty$, this implies the first bullet of Proposition 3.3. From this, it is easy to see that a horizontal curve intersects Y_g at a segment. This segment has its tangent space included in the cone field χ_g . By Proposition 8.5, this segment is $e^{\frac{c}{3}n_g}$ -expanded by f^{n_g} and its image has its tangent space in χ . This implies the last bullet of Proposition 3.3, and so this achieves its proof.

By Lemma 8.3, for every $x \in Y_g$, the \mathcal{F}_{n_g} -leaf \mathcal{C}_{n_g} of x is \sqrt{b} - C^1 -close to a component of a curve of the form $\{f_a(x) + y = cst\} \cap Y_e$. Moreover \mathcal{C}_{n_g} is $b^{k/2}$ contracted by f^k for every $k \leq n_g$.

If x belongs to $\partial^s Y_g$, then by the same Lemma, \mathcal{C}_{n_g} is $b^{n_g/2}$ - C^1 -close to the corresponding component of $\partial^s Y_g$. The latter is still $b^{k/2}$ contracted by f^k for every $k \leq n_g$. In this case, we chose this component for the curve \mathcal{C} requested by Lemma 3.9.2.

If $x \in Y_g$ belongs to $Y_g \setminus \partial^s Y_g$, either \mathcal{C}_{n_g} is included in Y_g , and so we put $\mathcal{C} = \mathcal{C}_{n_g}$ either \mathcal{C}_{n_g} intersects $\partial^s Y_g$. Then we can patch a segment of $\partial^s Y_g$ to a segment of \mathcal{C}_{n_g} in order to construct the curve \mathcal{C} claimed by Lemma 3.9.2.

It remains only the last statement of Lemma 3.9 to be proved. We just showed that $\partial^s Y^g$ is θ^{n_g} -small. Furthermore, as the tangent space of $\partial^u Y_g$ is in χ_g , by Proposition 8.5, the tangent space of $\partial^u Y^g$ is in χ . To show that $\partial^u Y^g$ consists of two flat curves, it remains only to show that these two curves have a very small curvature, which is an immediate consequence of (8.3) and of Lemma 2.4 of [YW01]. \square

Proof of Lemma 8.6. We already showed (\mathcal{PCE}^{n_g}) in (8.4). In order to show (\mathcal{PCE}^∞) , we are going to show that the points in $\partial^s Y_g$ are sent, a few iterates after n_g , into $\partial^s Y_e$ (which is in a local stable manifold of A).

Let us first regard a parabolic symbol $\Delta := \square_\pm(c_i - c_{i+1}) \in \mathfrak{A}$ and $c_{i+1} = c_i \star \alpha_{i+1}$.

In the proof of Proposition 5.9 of [Ber11] done in §14.2, equations (44) and (45) imply that for $z \in \partial^s Y_\Delta$ and $u \in \chi_\Delta$, the vector $Tf^{n_\Delta}(u)$ belongs to the “cone field of the canonical extension of α_{i+1} ”. This cone field satisfies h -times property by Proposition 5.9 of [Ber11] and is sent into χ by $Tf^{n_{\alpha_{i+1}}}$. Consequently for every $z \in \partial^s Y_\Delta$, $u \in \chi_\Delta(z)$, $k \leq n_{\alpha_{i+1}}$:

$$(8.5) \quad f^{n_\Delta + n_{\alpha_{i+1}}}(z) \in \partial^s Y_e, \quad T_z f^{n_\Delta + n_{\alpha_{i+1}}}(u) \in \chi \text{ and } \|T_z f^{n_\Delta + n_{\alpha_{i+1}}}(u)\| \geq e^{(n_{\alpha_{i+1}} - k)c/3} \|T_z f^{n_\Delta + k}(u)\|.$$

As A is a repelling fixed points, for every vector $u' \in \chi$ and $z' \in \partial^s Y_e$, it holds

$$\|T_{z'} f^k(u')\| \geq \|u'\|, \quad \forall k \geq 0,$$

and so it comes that for every $z \in \partial^s Y_\Delta$, $u \in \chi_\Delta(z)$:

$$(8.6) \quad \|T_z f^k(u)\| \geq e^{(n_\Delta + n_{\alpha_{i+1}})c/3} \|u\|, \quad \forall k \geq n_\Delta + n_{\alpha_{i+1}}.$$

We recall that by definition of common sequences, $n_{\alpha_{i+1}} \leq M + n_\Delta / \Xi$.

From 8.5 and 8.6, for every $k \geq n_\Delta$:

$$(8.7) \quad \|T_z f^k(u)\| \geq e^{-Mc^+} \|u\|.$$

We recall that $g = a_1 \cdots a_k$. By definition of $Y_{a_{j+1}}$, for every point $z \in \partial^s Y_g$, there exists $j < k$ such that $f^{n_{a_1 \cdots a_j}}(z)$ belongs to $\partial^s Y_{a_{j+1}}$. Furthermore, for every $u \in \chi(z)$, $T_z f^{n_{a_1 \cdots a_j}}(u)$ belongs to $\chi_{a_{j+1}}$ by (8.1). By (8.7), this implies that:

$$\|T_z f^k(u)\| \geq e^{-Mc^+} \|u\| \geq e^{-\Xi(M+1+k)c^+} \|u\| \quad \forall k \geq n_g.$$

The latter equation with (8.4) achieves the proof of this lemma. \square

Proof of Proposition 3.11. Let us suppose for the sake of contradiction that ν is ergodic and has one Lyapunov exponent non-negative. This implies that for ν -almost every point $z \in Y_e$, there exists a unit vector u and $N \geq 0$ such that:

$$(8.8) \quad \|T_z f^n(u)\| \geq e^{-cn}, \forall n \geq N.$$

We suppose for the sake of contradiction that z is not eventually $\sqrt{\Xi}$ -regular and $a_i(z) \neq \square$ for some i .

To simplify, we denote by $(a_i)_i$ the sequence $(a_i(z))_i$ associated to z . By replacing z by an iterate, we can suppose that $a_1 = \square$ and $a_2 \neq \square$.

Let $(i_j)_{j \geq 1}$ be the increasing sequence of integers defined by $a_k = \square$ iff $k = i_j$. Note that $i_1 = 1$.

Put $N_1 = 0$ and for $j \geq 2$, put $N_j := \sum_{i_{j-1} \leq l < i_j} n_{a_l}$. Let $n_j := N_1 + \cdots + N_j$ be the j^{th} -irregular return time and let $z_{n_j} := f^{n_j}(z)$ be the j^{th} irregular return of z .

Let us show by induction that for every j , $a_{i_{j+1}}$ is not \square . Let $j \geq 2$. If $a_{i_{j-1}} + 1 \neq \square$ then $z_{n_j} \in Y_{\square c}$ or $z_{n_j} \in Y_{\square_\delta(c_k - c_{k+1})}$ with $\delta \in \{\pm, b\}$ and $n_{c_k} + M + 1 > \Xi$ (and so $k > 1$). Thus the symbol $a_{i_{j+1}}$ is not \square .

The point z_{n_j} belongs to Y_a with $a \in \mathfrak{P}(t_j)$, with $t_j := \sharp \cdot a_{i_{j-1}+1} \cdots a_{i_j-1}$. The symbol a is of the form $\square_\delta(c_k - c_{k+1})$, $\delta \in \{\pm, b\}$. It satisfies:

$$n_a \geq M + 1 + \sqrt{\Xi}(N_j - M - 1)$$

As $N_j - M - 1 \geq 1$, it comes $n_a \geq \frac{\sqrt{\Xi}}{M+2} N_j$.

The \mathfrak{A} -spelling c_k of c_k is $\sqrt{\Xi}$ -regular, and so it is equal to the first symbols of $a_{i_{j+1}} \cdots a_{i_{j+1}-1}$. Thus it comes $N_{j+1} \geq n_a$ and:

$$(8.9) \quad N_{j+1} \geq n_a \geq \frac{\sqrt{\Xi}}{M+2} N_j \geq \left(\frac{\sqrt{\Xi}}{M+2} - 1 \right) N_j + N_j \geq \left(\frac{\sqrt{\Xi}}{M+2} - 1 \right) \sum_{l=1}^j N_l = \left(\frac{\sqrt{\Xi}}{M+2} - 1 \right) n_j$$

Take z , u , N be given by (8.8). Remember that $t_j := \sharp \cdot a_{i_{j-1}+1} \cdots a_{i_j-1}$. Put $u_k := T_z f^k(u)$, for $k \geq 0$.

Lemma 8.7. *For every $j \geq 2 + N$, there exist $z'_{n_j} \in S^{t_j}$ and a unit vector $u'_{n_j} \in T_{z'_{n_j}} S^{t_j}$ (i.e. tangent to S^{t_j} at z'_{n_j}) such that the angle between u'_{n_j} and u_{n_j} is $\theta^{n_j/5}$ -small and furthermore z_{n_j} and z'_{n_j} are $\theta^{n_j/2}$ -close.*

The proof of this lemma is postponed to the end.

We want to show that $\|u_{2n_j}\|$ is small with respect to e^{-2cn_j} when j is large. This would be a contradiction with inequality (8.8).

Put $w := u_{n_j}/\|u_{n_j}\|$ and $w' := u'_{n_j}/\|u'_{n_j}\|$. A classical computation gives:

$$\begin{aligned} \|T_{z_{n_j}} f^{n_j}(w) - T_{z'_{n_j}} f^{n_j}(w')\| &\leq \|T_{z_{n_j}} f^{n_j} - T_{z'_{n_j}} f^{n_j}\| + \|T_{z_{n_j}} f^{n_j}\| \cdot \|w - w'\| \leq n_j e^{2n_j c^+} \theta^{n_j/2} + e^{c^+ n_j} \theta^{n_j/5}. \\ (8.10) \quad &\Rightarrow \|T_{z_{n_j}} f^{n_j}(w)\| \leq \|T_{z'_{n_j}} f^{n_j}(w')\| + 2e^{c^+ n_j} \theta^{n_j/5}. \end{aligned}$$

On the other hand, the point z_{n_j} belongs to $Y_{\square} \cap f^{-M-1}(Y_{c_k})$ with $n_{c_k} + M + 1 \geq (\frac{\sqrt{\Xi}}{M+2} - 1)n_j$ by (8.9).

If c_k is the product of the pieces $\alpha_1 \star \dots \star \alpha_k$, then for $l \leq k$, we denote by c_l the product $\alpha_1 \star \dots \star \alpha_l$.

Let l be minimal such that $n_{c_l} + M + 1 \geq M \cdot n_j$. By the third item of the common sequence definition, Y_{c_l} is a neighborhood of Y_{c_k} sufficiently large in Y_e such that z'_{n_j} belongs to $Y_{\sigma_l} := Y_{\square} \cap f^{-M-1}(Y_{c_l})$.

By Proposition 14.3 of [Ber11], there exists a unique critical point \tilde{z} of S^{t_j} in Y_{σ_l} of order $M + 1 + n_{c_l}$. By Proposition 14.2 of [Ber11], every point $z'' \in Y_{\sigma_l}$ satisfies $(\mathcal{PC}\mathcal{E}^{n_{c_l}+M+1})$.

By Proposition 14.1 of [Ber11] and Lemma 8.2, there exists a splitting $w' =: w'_1 e_{n_j} + w'_2(0, 1)$, such that:

$$|w'_1| \leq (1 + \theta) + d(z'_{n_j}, \tilde{z}) \leq 2, \quad |w'_2| \leq 2(1 + \theta)d(z'_{n_j}, \tilde{z}) \text{ and } \|T_{z'_{n_j}} f^{n_j}(e_{n_j})\| \leq b^{n_j/2}.$$

The segment $S^{t_j} \cap Y_{\sigma_l}$ is an union of parabolic pieces. By Proposition 8.5, the length $S^{t_{n_j}} \cap Y_{\sigma_l}$ is less than $\sum_{k \geq M+1+n_{c_l}} e^{-kc/3}$ times the width of Y_e (which is less than 4). Thus:

$$|w'_2| \leq (1 + \theta) \frac{4e^{-\frac{c}{3}Mn_j}}{1 - e^{-c/3}}.$$

It comes that:

$$(8.11) \quad \|T f^{n_j}(w')\| \leq 2b^{n_j/2} + (1 + \theta) \frac{4e^{-\frac{c}{3}Mn_j}}{1 - e^{-c/3}} e^{n_j c^+} \ll e^{-4c^+ n_j}$$

Consequently:

$$\|u_{2n_j}\| \leq e^{n_j c^+} \|T f^{n_j}(w)\| \leq e^{n_j c^+} (n_j e^{n_j c^+} \theta^{n_j/5} + e^{-4c^+ n_j}) \ll e^{-2cn_j}.$$

This contradicts equation (8.8). Thus ν cannot have one exponent nonnegative. \square

Proof of Lemma 8.7. The point $z_{n_{j-1}+M+1} := f^{n_{j-1}+1}(z)$ is $i_j - i_{j-1} - 1$ -regular. Note that $g := a_{i_{j-1}+1} \cdot a_{i_{j-1}+2} \cdots a_{i_j-1}(z)$ is $\sqrt{\Xi}$ -regular and consists of the first letter of $\underline{a}(z_{n_{j-1}+M+1})$. In Lemma 3.9, we saw that $z_{n_{j-1}+M+1}$ belongs to a curve \mathcal{C} which satisfies the following properties:

- (i) For every $k \leq n_g = N_j - M - 1$, $\text{diam } f^k(\mathcal{C}) \leq \theta^k$.
- (ii) The curve \mathcal{C} intersects every flat stretched curve.
- (iii) The curve \mathcal{C} is included in Y_g .

By (ii), there exists a point $z' \in \mathcal{C} \cap S^\#$. By (iii), the point z' belongs to $S_g^\#$. Thus $z'_{n_j} := f^{n_g}(z')$ belongs to S^{t_j} . By (i), the distance between z'_{n_j} and z_{n_j} is less than θ^{n_g} . By (8.9), we have:

$$N_j + \frac{1}{\frac{\Xi}{M+2} - 1} N_j \geq n_j.$$

As $j \geq 2$ and as by (8.9) $N_j \geq (\sqrt{\Xi}/(M+2) - 1)n_{j-1} \gg 3M + 3$, it comes $N_j - M - 1 \geq 2N_j/3$ and so:

$$(8.12) \quad n_g = N_j - M - 1 > n_j/2.$$

It follows that the distance between z'_{n_j} and z_{n_j} is less than $\theta^{n_j/2}$.

Take any unitary vector u' tangent at z' to $S^\#$. Put $u'_{n_j} := T f^{n_g}(u')$. To evaluate the angle between u'_{n_j} and u_{n_j} , we regard the formula:

$$|\sin \angle(u'_{n_j}, u_{n_j})| = \frac{\|u'_{n_j} \times u_{n_j}\|}{\|u'_{n_j}\| \cdot \|u_{n_j}\|}$$

Put $n' := [n_g/2] + 1$. Let $x := f^{-n'}(z_{n_j})$ and $x' := f^{-n'}(z'_{n_j})$. Put also $u'_{n_j-n'} := (T_{x'} f^{n'})^{-1}(u'_{n_j})$. We have

$$\begin{aligned} |\sin \angle(u'_{n_j}, u_{n_j})| &= \frac{\|T_{x'} f^{n'} u'_{n_j-n'} \times T_x f^{n'} u_{n_j-n'}\|}{\|u'_{n_j}\| \cdot \|u_{n_j}\|} \\ &\leq \frac{|\det(T_x f^{n'})| \cdot \|u'_{n_j-n'} \times u_{n_j-n'}\|}{\|u'_{n_j}\| \cdot \|u_{n_j}\|} + \frac{\|T_x f^{n'} - T_{x'} f^{n'}\| \cdot \|u'_{n_j-n'}\|}{\|u'_{n_j}\|}. \end{aligned}$$

Let us study the first term of this sum. Since the determinant is less than b , $|\det(T_x f^{n'})| \leq b^{n'}$. By h -times property of g (Lemma 3.9.1), $\|u'_{n_j-n'}\|/\|u'_{n_j}\| \leq e^{-n'c/3}$. By inequality (8.8), as $n_j \geq j \geq N$:

$$\|u_{n_j}\| \geq e^{-cn_j} \|u_0\|; \quad \|u_{n_j-n'}\| \leq e^{c^+(n_j-n')} \|u_0\| \Rightarrow \frac{\|u_{n_j-n'}\|}{\|u_{n_j}\|} \leq e^{cn_j+c^+(n_j-n')} \leq e^{2c^+n_j}.$$

Consequently:

$$\frac{|\det(T_x f^{n'})| \cdot \|u'_{n_j-n'} \times u_{n_j-n'}\|}{\|u'_{n_j}\| \cdot \|u_{n_j}\|} \leq b^{n'} e^{2c^+n_j} e^{-n'c/3}.$$

Using again h -times property, the second term is bounded from above by $\|T_x f^{n'} - T_{x'} f^{n'}\| e^{-n'c/3}$. A classical computation gives, $\|T_x f^{n'} - T_{x'} f^{n'}\| \leq (n' + 1) e^{2n'c^+} \theta^{n'}$. Therefore:

$$|\sin \angle(u'_{n_j}, u_{n_j})| \leq b^{n'} e^{2c^+n_j} e^{-n'c/3} + (n' + 1) e^{n'c^+-n'c/3} \theta^{n'}.$$

By (8.12), it comes:

$$|\sin \angle(u'_{n_j}, u_{n_j})| \leq b^{n_j/4} e^{2c^+ n_j} + (n_j/4 + 1) e^{n_j/c^+/2 - n_j c/12} \theta^{n_j/4} \leq \theta^{n_j/5}.$$

□

Proof of Proposition 4.3. We are going to prove that for every invariant ergodic measure μ , the subsets $\cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$ and $\cup_{n \geq 0} f^n(\tilde{\mathcal{R}})$ are equal μ -almost everywhere.

The first subset contains clearly the second one. By ergodicity and invariance of the subset, we can suppose $\cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$ of full measure.

Let us show that μ -almost every point $x \in \cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$ is also in $\cup_{n \geq 0} f^n(\tilde{\mathcal{R}})$.

We observe that the measure of \mathcal{R} is positive, and so by Lemma 3.9, one Lyapunov exponent of μ is greater than $c/3$. Thus for μ -almost every point $x \in \cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$, there exists a vector u such that:

$$(8.13) \quad \frac{1}{n} \log \|T_x f^{-n}(u)\| \leq -\frac{c}{4}$$

for every $n \geq 0$ large enough. Note that Inequation 8.13 for every $n \geq 0$ (and not only n large enough) is true for a μ -positive set.

As $\cup_{n \geq 0} f^n(\tilde{\mathcal{R}})$ is invariant and μ ergodic, it is sufficient to show that its measure is positive.

Consequently it is sufficient to prove that every $x \in \cap_{N \geq 0} \cup_{n \geq N} f^n(\mathcal{R})$ satisfying Inequation 8.13 for every $n \geq 0$ is in $\cup_{n \geq 0} f^n(\tilde{\mathcal{R}})$.

Let n be large and such that $x \in f^n(\mathcal{R})$. Suppose that $z^n := f^{-n}(x)$ belongs to Y_c . Let $\underline{a}^n = (a_i^n)_i := \underline{a}(z^n)$.

Remark that \underline{a}^n belongs to R and so starts by a work $g \in \mathfrak{B}$. A first problem is that *a priori* g_1 could be larger than n whatever n is...

That is why we shall find $N' \geq 0$ such that there exist $q \geq 0$ and n arbitrarily large satisfying that \underline{a}^n belongs to $g_1 \cdots g_q \cdot \mathfrak{A}^{\mathbb{N}}$ and $n_{g_1 \cdots g_q} \leq n \leq n_{g_1 \cdots g_q} + N'$.

This implies that x belongs to $\cup_{N' \geq n \geq 0} f^n(F^q(\mathcal{R}))$ for every q , and so to $\cup_{N' \geq n \geq 0} f^n(\tilde{\mathcal{R}})$.

The definition of N' depends on the following integer N . By convergence of the Lyapunov exponent, there exists $N \geq 2\Xi^2$ such that for every $k \geq k' \geq N$:

$$(8.14) \quad -\frac{c}{10} \leq \frac{1}{k} \log \|T_x f^{-k}(u)\| - \frac{1}{k'} \log \|T_x f^{-k'}(u)\| \leq \frac{c}{10}$$

Let us suppose $n \geq N$, and take z and \underline{a}^n be defined as above.

For every $p \in [0, n]$, let $a_{i_p}^n$ be the symbol such that $-n + n_{a_1^n \cdots a_{i_p}^n} < -p \leq -n + n_{a_1^n \cdots a_{i_p}^n}$.

Below we show the two following:

Claim 8.8. *For every $p \geq N$, the symbol $a_{i_p}^n$ has an order less than $p/2$.*

This claim is sufficient to conclude. Indeed, take a strictly increasing sequence $(n_j)_{j \geq 0}$ such that every n_j is greater than N and $x \in f^{n_j}(\mathcal{R})$.

From Claim 8.8 at every $p \in [N, n_j]$, by diagonal extraction, we can suppose that $a_{i_p} := a_{i_p}^{n_j}$ does not depend on j . Put $m_j := i_{n_j}$. The word $a_{m_j} \cdot a_{m_j-1} \cdots a_{i_N}$ is Ξ -regular. Also x belongs to

$f^{n_j}(Y_{a_{m_j} \cdot a_{m_j-1} \cdots a_{i_N}})$. Observe that m_j is greater than the order of $a_{m_j} \cdot a_{m_j-1} \cdots a_{i_N}$. As $(m_j)_j$ is increasing, by definition of \mathfrak{B} , there exist $q \geq j$ and $(g_k)_{1 \leq k \leq q} \in \mathfrak{B}^q$ such that:

$$a_{m_j} \cdot a_{m_j-1} \cdots a_{i_N} = g_q \cdots g_2 \cdot g_1 \cdot a_{m_0} \cdots a_{i_N}.$$

Therefore x belongs to $\cup_{0 \leq n \leq N'} f^n(F^q(\mathcal{R}))$ with $N' := N + n_{a_{m_0} \cdots a_{i_N}}$. \square

Proof of Claim 8.8. Let us fix n , and for the sake of simplicity, let us write $\underline{a} = (a_i)_i$ instead of $\underline{a}^n = (a_i^n)_n$ and z instead of z^n . We suppose that $n \geq N \geq 2\Xi^2$.

The sequence \underline{a} belongs to R , thus for $p \in \left[n - \frac{n/3-M}{\Xi}, n\right]$, the symbol a_{i_p} has an order at most $M + \Xi \frac{n/3-M}{\Xi} = n/3 \leq p/2$.

Let $p \in \left[N, n - \frac{n/3-M}{\Xi}\right)$. We remark that $\underline{a}' = a_1 \cdots a_{i_{p-1}}$ belongs to \mathfrak{R} . Observe that $\underline{a} = \underline{a}' \cdot a_{i_p} \cdot a_{i_{p+1}} \cdots$. We want to show that the order of a_{i_p} is at most $p/2$. Let $d := a_{i_p}$.

Let z' be the intersection point of $W_{\underline{a}}^s$ with S^\sharp . Put $v := Tf^{-n}(u)$. Let v' be a unit vector of $T_{z'}S^\sharp$.

For every $k \geq 0$, put $z_k := f^k(z)$, $z'_k := f^k(z')$, $v_k := Tf^k(v)$ and $v'_k := Tf^k(v')$.

Lemma 8.9. *The points $z_{n_{\underline{a}'}}$ and $z'_{n_{\underline{a}'}}$ are $\theta^{n_{\underline{a}'}}$ -close, and $\sin \angle(v_{n_{\underline{a}'}} , v'_{n_{\underline{a}'}})$ is $\theta^{n_{\underline{a}'}/3}$ -small.*

The proof of this lemma follows the same argument as Lemma 8.7 and is done below.

As for (8.10), it comes for $k \leq n_d$:

$$\begin{aligned} \frac{\|v_{n_{\underline{a}'}+k}\|}{\|v_{n_{\underline{a}'}}\|} - \frac{\|v'_{n_{\underline{a}'}+k}\|}{\|v'_{n_{\underline{a}'}}\|} &\leq \|T_{z_{n_{\underline{a}'}}} f^k - T_{z'_{n_{\underline{a}'}}} f^k\| + \|T_{z_{n_{\underline{a}'}}} f^k\| \cdot \|v_{n_{\underline{a}'}} - v'_{n_{\underline{a}'}}\| \leq (k+1)e^{2c^+(k+1)} \theta^{n_{\underline{a}'}} + e^{c^+k} \theta^{n_{\underline{a}'}/3} \\ (8.15) \quad &\Rightarrow \frac{\|v_{n_{\underline{a}'}+k}\|}{\|v_{n_{\underline{a}'}}} - \frac{\|v'_{n_{\underline{a}'}+k}\|}{\|v'_{n_{\underline{a}'}}} \leq 2e^{c^+k} \theta^{n_{\underline{a}'}/3} \end{aligned}$$

On the other hand, as for (8.11), it comes for $k \leq n_d$:

$$(8.16) \quad \frac{\|T_{z'} f^{n_{\underline{a}'}+k}(v')\|}{\|T_{z'} f^{n_{\underline{a}'}}(v')\|} \leq 2b^{k/2} + (1+\theta) \frac{4e^{-\frac{c}{3}n_d}}{1-e^{-c/3}} e^{c^+k}$$

For $k \geq 0$, put $u_{-k} := T_x f^{-k}(u)$. Put $m := n - n_{\underline{a}'}$. Observe that:

$$(8.17) \quad n_{\underline{a}'} \leq \frac{n/3-M}{\Xi} \leq M + (\Xi+1)n_{\underline{a}'} \Rightarrow m = n - n_{\underline{a}'} \leq M + \Xi(M + (\Xi+1)n_{\underline{a}'}) - n_{\underline{a}'}$$

We remark that $m \geq p \geq N$. Observe that $v = u_{-n}$ and $v_{n_{\underline{a}'}} = u_{-m}$. From (8.15) and (8.16), it holds for every $k \leq n_d$:

$$\frac{\|u_{-m+k}\|}{\|u_{-m}\|} \leq 2b^{k/2} + (1+\theta) \frac{4e^{-\frac{c}{3}n_d}}{1-e^{-c/3}} e^{c^+k} + 2e^{c^+k} \theta^{n_{\underline{a}'}}$$

Let us suppose for the sake of contradiction that $n_d \geq m/3$. Then with $k = [m/M]$, by (8.17):

$$\frac{\|u_{-m+[m/M]}\|}{\|u_{-m}\|} \leq 2b^{m/2M-1} + (1+\theta) \frac{4e^{-\frac{c}{3}n_d}}{1-e^{-c/3}} e^{c^+m/M} + 2e^{c^+[m/M]} \theta^{\frac{m}{2\Xi^2}} \leq e^{-\frac{c}{10}m}$$

Taking the logarithm and dividing by m , this implies since $\log \|u_{-m+[m/M]}\|$ is negative:

$$\frac{1}{m - [m/M]} \log \|u_{-m+[m/M]}\| - \frac{1}{m} \log \|u_{-m}\| < \frac{1}{m} \log \|u_{-m+[m/M]}\| - \frac{1}{m} \log \|u_{-m}\| \leq -\frac{c}{10}$$

This contradicts (8.14). Thus $n_d \leq m/3$ and as $m - n_d \leq p$, it holds $n_d \leq p/2$. \square

Proof of lemma 8.9. By Lemma 3.9, the points z_k and z'_k are θ^k -close and so $z_{n_{\underline{a}'}}$ and $z'_{n_{\underline{a}'}}$ are $\theta^{n_{\underline{a}'}}$ -close.

Let us bound from above the angle:

$$|\sin \angle(v_{n_{\underline{a}'}}', v'_{n_{\underline{a}'}})| = \frac{\|v_{n_{\underline{a}'}}' \times v'_{n_{\underline{a}'}}'\|}{\|v_{n_{\underline{a}'}}'\| \cdot \|v'_{n_{\underline{a}'}}'\|}$$

Put $n' := [n_{\underline{a}'}/2] + 1$. Let $y := f^{n_{\underline{a}'}-n'}(z)$ and $y' := f^{n_{\underline{a}'}-n'}(z')$. We have

$$\begin{aligned} |\sin \angle(v_{n_{\underline{a}'}}', v'_{n_{\underline{a}'}}')| &= \frac{\|T_y f^{n'} v_{n_{\underline{a}'}-n'} \times T_{y'} f^{n'} v'_{n_{\underline{a}'}-n'}\|}{\|v_{n_{\underline{a}'}}'\| \cdot \|v'_{n_{\underline{a}'}}'\|} \\ &\leq \frac{|\det(T_y f^{n'})| \cdot \|v_{n_{\underline{a}'}-n'} \times v'_{n_{\underline{a}'}-n'}\|}{\|v_{n_{\underline{a}'}}'\| \cdot \|v'_{n_{\underline{a}'}}'\|} + \frac{\|T_y f^{n'} - T_{y'} f^{n'}\| \cdot \|v'_{n_{\underline{a}'}-n'}\|}{\|v'_{n_{\underline{a}'}}'\|}. \end{aligned}$$

By h -times property of \underline{a}' (Lemma 3.9.1), $\|v'_{n_{\underline{a}'}-n'}\|/\|v'_{n_{\underline{a}'}}'\| \leq e^{-n'c/3}$.

Thus the second term is bounded from above by $\|T_x f^{n'} - T_{x'} f^{n'}\| e^{-n'c/3}$. The same classical computation as for (8.10) gives, $\|T_y f^{n'} - T_{y'} f^{n'}\| \leq n' e^{2n'c^+} \theta^{n'}$. As n' is greater than $n_{\underline{a}'}/2$, the second term is less than $\theta^{n_{\underline{a}'}/3}/2$.

Let us study the first term of this sum. Since the determinant is less than b , $|\det(T_y f^{n'})| \leq b^{n'}$.

By inequality (8.13):

$$\|v_{n_{\underline{a}'}-n'}\| \leq e^{-\frac{c}{4}(n-(n_{\underline{a}'}-n'))}, \quad \|v_{n_{\underline{a}'}}'\| \geq e^{(n-n_{\underline{a}'})c^+} \Rightarrow \frac{\|v_{n_{\underline{a}'}-n'}\|}{\|v_{n_{\underline{a}'}}'\|} \leq e^{(n_{\underline{a}'}-n-n')\frac{c}{4}+(n_{\underline{a}'}-n)c^+} \leq e^{2n_{\underline{a}'}c^+}.$$

Consequently:

$$\frac{|\det(T_{y'} f^{n'})| \cdot \|v'_{n_{\underline{a}'}-n'} \times v_{n_{\underline{a}'}-n'}\|}{\|v'_{n_{\underline{a}'}}'\| \cdot \|v_{n_{\underline{a}'}}'\|} \leq b^{n'} e^{2c^+ n_{\underline{a}'}} e^{-n'c/3}.$$

As n' is greater than $n_{\underline{a}'}/2$, the first term is less than $\theta^{n_{\underline{a}'}/3}/2$.

Therefore:

$$|\sin \angle(v'_{n_{\underline{a}'}}', v_{n_{\underline{a}'}})| \leq \theta^{n_{\underline{a}'}/5}.$$

\square

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